

# Eigenvalue multiplicity and volume growth

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## Abstract

Let  $G$  be a finite group with symmetric generating set  $S$ , and let  $c = \max_{R>0} \frac{|B(2R)|}{|B(R)|}$  be the doubling constant of the corresponding Cayley graph, where  $B(R)$  denotes an  $R$ -ball in the word-metric with respect to  $S$ . We show that the multiplicity of the  $k$ th eigenvalue of the Laplacian on the Cayley graph of  $G$  is bounded by a function of only  $c$  and  $k$ . More specifically, the multiplicity is at most  $\exp(O(\log c)(\log c + \log k))$ .

Similarly, if  $X$  is a compact,  $n$ -dimensional Riemannian manifold with non-negative Ricci curvature, then the multiplicity of the  $k$ th eigenvalue of the Laplace-Beltrami operator on  $X$  is at most  $\exp(O(n)(n + \log k))$ .

The first result (for  $k = 2$ ) yields the following group-theoretic application. There exists a normal subgroup  $N$  of  $G$ , with  $[G : N] \leq \alpha(c)$ , and such that  $N$  admits a homomorphism onto  $\mathbb{Z}_M$ , where  $M \geq |G|^{\delta(c)}$  and

$$\begin{aligned}\alpha(c) &\leq O(h)^{h^2} \\ \delta(c) &\geq \frac{1}{O(h \log c)},\end{aligned}$$

where  $h \leq \exp((\log c)^2)$ . This is an effective, finitary analog of a theorem of Gromov which states that every infinite group of polynomial growth has a subgroup of finite index which admits a homomorphism onto  $\mathbb{Z}$ .

This addresses a question of Trevisan, and is proved by scaling down Kleiner's proof of Gromov's theorem. In particular, we replace the space of harmonic functions of fixed polynomial growth by the second eigenspace of the Laplacian on the Cayley graph of  $G$ .

## 1 Introduction

Let  $G$  be a finitely generated group with finite, symmetric generating set  $S$ . The Cayley graph  $\text{Cay}(G; S)$  is an undirected  $|S|$ -regular graph with vertex set  $G$  and an edge  $\{u, v\}$  whenever  $u = vs$  for some  $s \in S$ . We equip  $G$  with the natural word metric, which is also the shortest-path metric on  $\text{Cay}(G; S)$ . Letting  $B(R)$  be the closed ball of radius  $R$  about  $e \in G$ , one says that  $G$  has *polynomial growth* if there exists a number  $m \in \mathbb{N}$  such that

$$\lim_{R \rightarrow \infty} \frac{|B(R)|}{R^m} < \infty.$$

It is easy to see that this property is independent of the choice of finite generating set  $S$ .

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In a classical paper [10], Gromov proved that a group has polynomial growth if and only if it contains a nilpotent subgroup of finite index. The sufficiency part was proved earlier by Wolf [16]. It is natural to ask about similar phenomenon holds in *finite groups*. Of course, every finite group has polynomial growth trivially, so even formulating a similar question is not straightforward. As Gromov points out [10], by a compactness argument, one only needs  $|B(R)| \leq CR^m$  to hold for  $R \leq R_0$ , for some  $R_0 = R_0(C, |S|, m)$ . Thus one can formulate a version of Gromov's theorem for finite groups. However, there are no effective estimates known for  $R_0$ . Furthermore, Gromov's proof relies on a limiting procedure which is again trivial for finite groups.

Recently, Kleiner [13] gave a new proof of Gromov's theorem that avoids the limiting procedure, and in particular avoids the use of the Yamabe-Montgomery-Zippin structure theory [14] to classify the limit objects. The main step of Kleiner's proof lies in showing that the space of harmonic functions of fixed polynomial growth is finite-dimensional on an infinite group  $G$  of polynomial growth. Such a result follows from the work of Colding and Minicozzi [8], but their proof uses Gromov's theorem, whereas Kleiner is able to obtain the result essentially from scratch, based on a new scale-dependent Poincaré inequality for bounded-degree graphs.

Again, the connection with finite groups is lacking: Every harmonic function on a finite graph is constant. In the present work, we show that one can obtain some *effective* partial analogs of Gromov's theorem for finite groups by following Kleiner's general outline, but replacing the space of harmonic functions of fixed polynomial growth with the second eigenspace of the discrete Laplacian on  $\text{Cay}(G; S)$ .

We recall the following two theorems of Gromov, which capture the essential move from a geometric condition (polynomial volume growth of balls) on an infinite group  $G$ , to a conclusion about its algebraic structure.

**Theorem 1.1** (Gromov [10]). *If  $G$  is an infinite group of polynomial growth, the following holds.*

1.  $G$  admits a finite-dimensional linear representation  $\rho : G \rightarrow GL_n(\mathbb{C})$  such that  $\rho(G)$  is infinite.
2.  $G$  contains a normal subgroup  $N$ , with  $[G : N] = O(1)$ , and such that  $N$  admits a homomorphism onto  $\mathbb{Z}$ .

In fact, by the simplifications of Tits (in Appendix A.2 of [10]), Gromov's theorem follows fairly easily using an induction on (2). After seeing Kleiner's proof, Luca Trevisan asked whether there is a quantitative analog of part (1) of Theorem 1.1 for finite groups. We prove the following.

**Theorem 1.2.** *Let  $G$  be a finite group. For any symmetric generating set  $S$ , define*

$$c = c_{G;S} = \max_{R \geq 0} \frac{|B(2R)|}{|B(R)|},$$

where  $B(\cdot)$  is a closed ball in  $\text{Cay}(G; S)$ . Then the following holds.

1. There is a linear representation  $\rho : G \rightarrow GL(\mathbb{R}^k)$ , where  $k \leq \exp(O(\log c)^2)$ , and  $|\rho(G)| \geq c^{-O(1)}|G|^{1/\log_2(c)}$ .
2. There is a normal subgroup  $N \leq G$ , with  $[G : N] = O(k)^{k^2}$ , and  $N$  admits a homomorphism onto  $\mathbb{Z}_M$ , where  $M \geq c^{-O(1)}|G|^{1/(k \log_2 c)}$ .

Observe that we have assumed a bound on the ratios  $|B(2R)|/|B(R)|$ , which is stronger than an assumption of the form

$$|B(R)| \leq CR^m \quad \text{for some } C, m > 0. \quad (1)$$

The latter type of condition seems far more unwieldy in the setting of finite groups. By making such an assumption, we completely bypass a “scale selection” argument, and the delicacy required by Kleiner’s approach (which has to perform many steps of the proof using only the geometry at a single scale). All of our arguments can be carried out at a single scale (see, e.g. the Reverse Poincaré Inequality for graphs in Section 3.1), but it is not clear whether there is an appropriate, effective scale selection procedure in the finite case, and we leave the extension of Theorem 1.2 to a bounded growth condition like (1) as an interesting open question.

### 1.1 Proof outline and eigenvalue multiplicity

Our proof of Theorem 1.2 proceeds along the following lines. Given an undirected  $d$ -regular graph  $H = (V, E)$ , one defines the *discrete Laplacian on  $H$*  as the operator  $\Delta : L^2(V) \rightarrow L^2(V)$  given by  $\Delta(f)(x) = f(x) - \frac{1}{d} \sum_{y:\{x,y\} \in E} f(y)$ . The eigenvalues of  $\Delta$  are non-negative and can be ordered  $\lambda_1 = 0 \leq \lambda_2 \leq \dots \leq \lambda_n$ , where  $n = |V|$ . The *second eigenspace of  $\Delta$*  is the subspace  $W_2 \subseteq L^2(V)$  given by  $W_2 = \{f \in L^2(V) : \Delta f = \lambda_2 f\}$ . Finally, the well-known (geometric) *multiplicity of  $\lambda_2$*  is precisely  $\dim(W_2)$ .

In Section 3, we use the approach of Colding and Minicozzi [8] and Kleiner [13] to argue that  $\dim(W_2) = O(1)$ , whenever  $c_H = \max_{x \in V, R \geq 0} \frac{|B(x, 2R)|}{|B(x, R)|} = O(1)$ , and  $H$  satisfies a certain Poincaré inequality. At the heart of the proof lies the intuition that functions in  $W_2$  are the “most harmonic-like” functions on  $H$  which are orthogonal to the constant functions. Carrying this out requires precise quantitative control on the eigenvalues of  $H$  in terms of  $c_H$ , which we obtain in Section 3.2.

Now, consider  $H = \text{Cay}(G; S)$  for some finite group  $G$ , and the natural action of  $G$  on  $f \in L^2(G)$  given by  $gf(x) = f(g^{-1}x)$ . It is easy to see that this action commutes with the action of the Laplacian, hence  $W_2$  is an invariant subspace. Since  $\dim(W_2) = O(1)$ , we will have achieved Theorem 1.2(1) as long as the image of the action is large. In Section 4, we show that if the image of the action is small, then we can pass to a small quotient group, and that  $f$  pushes down to an eigenfunction on the quotient. This allows us to bound  $\lambda_2$  on the quotient group in terms of  $\lambda_2$  on  $G$ . But  $\lambda_2$  on a small, connected graph cannot be too close to zero by the discrete Cheeger inequality. In this way, we arrive at a contradiction if the image of the action is too small. Theorem 1.2(2) is then a simple corollary of Theorem 1.2(1), using a theorem of Jordan on finite linear groups.

**Higher eigenvalues and non-negatively curved manifolds.** In fact, the techniques of Section 3 give bounds on the multiplicity of higher eigenvalues of the Laplacian as well, and the graph proof extends rather easily to bounding the eigenvalues of the Laplace-Beltrami operator on Riemannian manifolds of non-negative Ricci curvature.

Cheng [7] proved that the multiplicity of the  $k$ th eigenvalue of a compact Riemannian surface of genus  $g$  grows like  $O(g + k + 1)^2$ . Besson later showed [3] that the multiplicity of the first non-zero eigenvalue is only  $O(g + 1)$ . We refer to the book of Schoen and Yau [15, Ch. 3] for further discussion of eigenvalue problems on manifolds. In Section 3, we prove a bound on the multiplicity of the  $k$ th smallest non-zero eigenvalue of the Laplace-Beltrami operator on compact Riemannian manifolds with non-negative Ricci curvature. In particular, the multiplicity is bounded

by a function depending only on  $k$  and the dimension. The main additional fact we require is an eigenvalue estimate of Cheng [6] in this setting.

## 2 Preliminaries

### 2.1 Notation

For  $N \in \mathbb{N}$ , we write  $[N]$  for  $\{1, 2, \dots, N\}$ .

Given two expressions  $E$  and  $E'$  (possibly depending on a number of parameters), we write  $E = O(E')$  to mean that  $E \leq CE'$  for some constant  $C > 0$  which is independent of the parameters. Similarly,  $E = \Omega(E')$  implies that  $E \geq CE'$  for some  $C > 0$ . We also write  $E \lesssim E'$  as a synonym for  $E = O(E')$ . Finally, we write  $E \approx E'$  to denote the conjunction of  $E \lesssim E'$  and  $E \gtrsim E'$ .

In a metric space  $(X, d)$ , for a point  $x \in X$ , we use  $B(x, R) = \{y \in X : d(x, y) \leq R\}$  to denote the closed ball in  $X$  about  $x$ .

### 2.2 Laplacians, eigenvalues, and the Poincaré inequality

Let  $(X, \text{dist}, \mu)$  be a metric-measure space. Throughout the paper, we will be in one of the following two situations.

- (G)  $X$  is a finite, connected, undirected  $d$ -regular graph,  $\text{dist}$  is the shortest-path metric, and  $\mu$  is the counting measure. In this case, we let  $E(X)$  denote the edge set of  $X$ , and we write  $y \sim x$  to denote  $\{x, y\} \in E(X)$ .
- (M)  $X$  is a compact  $n$ -dimensional Riemannian manifold without boundary,  $\text{dist}$  is the Riemannian distance, and  $\mu$  is the Riemannian volume.

Since the proofs of Section 3 proceed virtually identically in both cases, we collect here some common notation. We define  $\|f\|_2 = (\int f^2 d\mu)^{1/2}$  for a function  $f : X \rightarrow \mathbb{R}$ , and let  $L^2(X) = L^2(X, \mu)$  be the Hilbert space of scalar functions for which  $\|\cdot\|_2$  is bounded. In the graph setting, we define the gradient by  $[\nabla f](x) = \frac{1}{\sqrt{2d}}(f(x) - f(y_1), \dots, f(x) - f(y_d))$ , where  $y_1, \dots, y_d$  enumerate the neighbors of  $x \in X$ . The actual order of enumeration is unimportant as we will be primarily concerned with the expression  $|\nabla f(x)|^2 = \frac{1}{2d} \sum_{y: y \sim x} |f(x) - f(y)|^2$ .

We define the Sobolev space

$$L_1^2(X) = \left\{ f : \int f^2 d\mu + \int |\nabla f|^2 d\mu < \infty \right\} \subseteq L^2(X).$$

Now we proceed to define the Laplacian  $\Delta : L_1^2(X) \rightarrow L_1^2(X)$ .

1. In the graph setting,  $[\Delta f](x) = f(x) - \frac{1}{d} \sum_{y: y \sim x} f(y)$ .
2. In the Riemannian setting,  $\Delta$  is the Laplace-Beltrami operator.

It is well-known that in both our settings,  $\Delta$  is a self-adjoint operator on  $L_1^2(X)$  with discrete eigenvalues  $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ . In the graph case, this sequence terminates with  $\lambda_{|X|}$ . (Note

that we have used the graph-theoretic convention for numbering the eigenvalues; in the Riemannian setting, our  $\lambda_1$  is usually written as  $\lambda_0$ .)

We define the  $k$ th eigenspace by

$$W_k = \{\varphi \in L_1^2(X) : \Delta\varphi = \lambda_k\varphi\}$$

in setting (G), and

$$W_k = \{\varphi \in L_1^2(X) : \Delta\varphi + \lambda_k\varphi = 0\},$$

in setting (M). The *multiplicity* of  $\lambda_k$  is defined as  $m_k = \dim(W_k)$ . Observe the difference in sign conventions, which will not disturb us since we interact with  $\Delta$  through the following two facts.

First, if  $\lambda$  is an eigenvalue of  $\Delta$  with corresponding eigenfunction  $\varphi : X \rightarrow \mathbb{R}$ , then

$$\int |\nabla\varphi|^2 d\mu = \lambda \int \varphi^2 d\mu. \quad (2)$$

Secondly, by the min-max principle, if we have functions  $f_1, f_2, \dots, f_k : X \rightarrow \mathbb{R}$  which have mutually disjoint supports (and are thus linearly independent), then we have the bound

$$\lambda_k \leq \max_{i=1, \dots, k} \frac{\int |\nabla f_i|^2 d\mu}{\int (f_i - \bar{f}_i)^2 d\mu}, \quad (3)$$

where  $\bar{f}_i = \frac{1}{\mu(X)} \int f_i d\mu$ . In the case  $k = 2$ , we actually need only a single test function  $f_1 : X \rightarrow \mathbb{R}$  in (3), since clearly  $f_1 - \bar{f}_1$  is orthogonal to every constant function.

**The doubling condition.** We define  $c_X = \sup \left\{ \frac{\mu(B(x, 2R))}{\mu(B(x, R))} : x \in X, R > 0 \right\}$ . Without loss of generality, and for the sake of simplicity, we will assume that  $c_X \geq 2$  throughout. The next theorem follows from standard volume comparison theorems (see, e.g. [12]).

**Theorem 2.1.** *In the setting (M), if  $X$  has non-negative Ricci curvature, then  $c_X \leq 2^n$ .*

The following two facts are straightforward.

**Fact 2.2.** *For every  $\varepsilon, R > 0$ , every ball of radius  $R$  in  $X$  can be covered by  $c_X^{O(\log(\varepsilon^{-1}))}$  balls of radius  $\varepsilon R$ .*

**Fact 2.3.** *If  $\mathcal{B} = \{B_1, \dots, B_M\}$  is a disjoint collection of closed balls of radius  $R$ , then the intersection multiplicity of  $3\mathcal{B} = \{3B_1, \dots, 3B_M\}$  is at most  $c_X^{O(1)}$ .*

**A Poincaré inequality.** Finally, we define  $P_X$  as the infimum over all numbers  $P$  for which the following holds: For every  $R \geq 0$ ,  $x \in X$ , and  $f : B(x, 3R) \rightarrow \mathbb{R}$ ,

$$\int_{B(x, R)} |f - \bar{f}_R|^2 d\mu \leq PR^2 \int_{B(x, 3R)} |\nabla f|^2 d\mu, \quad (4)$$

where  $\bar{f}_R = \frac{1}{\mu(B(x, R))} \int_{B(x, R)} f d\mu$ .

We recall the following two known results about the relationship between  $P_X$  and  $c_X$ .

**Theorem 2.4** (Kleiner and Saloff-Coste [13]). *In the setting (G), if  $X$  is additionally a Cayley graph, then  $P_X \lesssim c_X^3$ .*

**Theorem 2.5** (Buser [4]). *In the setting (M), if  $X$  has non-negative Ricci curvature, then  $P_X \lesssim c_X$ .*

### 3 Eigenvalue multiplicity on doubling spaces

In this section, we prove the following.

**Theorem 3.1.** *In both settings (G) and (M), the multiplicity  $m_k$  of the  $k$ th eigenvalue of the Laplacian on  $X$  satisfies*

$$\begin{aligned} m_2 &\leq c_X^{O(\log P_X + \log c_X)}, \quad \text{and} \\ m_k &\leq c_X^{O(\log P_X + c_X \log k)} \quad \text{for } k \geq 3. \end{aligned}$$

If in the setting (G),  $X$  is a Cayley graph, then for  $k \geq 2$ ,

$$m_k \leq \exp(O(\log c_X)(\log c_X + \log k)) \quad (5)$$

If in the setting (M),  $X$  additionally has non-negative Ricci curvature, then for  $k \geq 2$ ,

$$m_k \leq \exp(O(n^2 + n \log k)) \quad (6)$$

We will require the following eigenvalue bounds.

**Theorem 3.2** (Cheng [6]). *In setting (M), if  $X$  also has non-negative Ricci curvature, then the  $k$ th eigenvalue of the Laplacian on  $X$  satisfies*

$$\lambda_k \lesssim \frac{k^2 n^2}{\text{diam}(X)^2}.$$

Cheng's result is proved via comparison to a model space of constant sectional curvature. In general, we can prove a weaker bound under just a doubling assumption. The proof is deferred to Section 3.2.

**Theorem 3.3.** *In both settings (G) and (M), the following is true. The  $k$ th eigenvalue of the Laplacian on  $X$  satisfies*

$$\begin{aligned} \lambda_2 &\leq \frac{c_X^{O(1)}}{\text{diam}(X)^2}, \quad \text{and} \\ \lambda_k &\leq \frac{k^{O(c_X)}}{\text{diam}(X)^2} \quad \text{for } k \geq 3. \end{aligned}$$

If, in addition, for every  $x, y \in X$  and  $R \geq 0$ , we have  $\mu(B(x, R)) = \mu(B(y, R))$ , then one obtains the estimate

$$\lambda_k \lesssim \frac{k^2 (\log c_X)^2}{\text{diam}(X)^2}, \quad (7)$$

for all  $k \geq 2$ .

We proceed to the proof of the theorem.

*Proof of Theorem 3.1.* Let  $D = \text{diam}(X)$ , and let  $\mathcal{B} = \{B_1, B_2, \dots, B_M\}$  be a cover of  $X$  of minimal size by balls of radius  $\delta D$ , for some  $\delta > 0$  to be chosen later. By the doubling property (and Fact 2.2), we have  $M \leq c_X^{O(\log(\delta^{-1}))}$ .

Let  $W_k$  be the  $k$ th eigenspace of the Laplacian, and define the linear map  $\Phi : W_k \rightarrow \mathbb{R}^M$  by  $\Phi_j(\varphi) = \frac{1}{\mu(B_j)} \int_{B_j} \varphi d\mu$ . Our goal will be to show that for  $\delta > 0$  small enough,  $\Phi$  is injective, and thus  $\dim(W_k) \leq M$ .

**Lemma 3.4.** *If  $\varphi : X \rightarrow \mathbb{R}$  is a non-zero eigenfunction of the Laplacian with eigenvalue  $\lambda \neq 0$ , and  $\Phi(\varphi) = 0$ , then*

$$\lambda^{-1} \lesssim c_X^{O(1)} P_X(\delta D)^2.$$

*Proof.* Using  $\Phi_j(\varphi) = 0$  for every  $j \in [M]$ , and the Poincaré inequality (4), we write

$$\int \varphi^2 d\mu \leq \sum_{j=1}^M \int_{B_j} \varphi^2 d\mu \lesssim P_X(\delta D)^2 \sum_{j=1}^M \int_{3B_j} |\nabla \varphi|^2 d\mu.$$

Also,

$$\sum_{j=1}^M \int_{3B_j} |\nabla \varphi|^2 d\mu \leq \mathcal{M}(3\mathcal{B}) \int |\nabla \varphi|^2 d\mu,$$

where  $\mathcal{M}(3\mathcal{B}) = \max_{x \in V} \#\{j \in [M] : x \in 3B_j\} \leq c_X^{O(1)}$  is the intersection multiplicity of  $3\mathcal{B}$  (by Fact 2.3). Combining these two inequalities and using (2) yields

$$\int \varphi^2 d\mu \lesssim c_X^{O(1)} P_X(\delta D)^2 \int |\nabla \varphi|^2 d\mu \lesssim c_X^{O(1)} P_X(\delta D)^2 \lambda \int \varphi^2 d\mu$$

which gives the desired conclusion.  $\square$

Now suppose that  $\varphi \in W_k$  and  $\Phi(\varphi) = 0$ . If  $\varphi \neq 0$ , then by Lemma 3.4, we have

$$\lambda_k \gtrsim \frac{c_X^{-O(1)} P_X^{-1}}{\delta^2 \text{diam}(X)^2}.$$

Choosing  $\delta > 0$  small enough contradicts Theorem 3.2 or Theorem 3.3, depending upon the assumption. It follows that  $\dim(W_k) \leq M \leq c_X^{O(\log(\delta^{-1}))}$ , yielding the desired bounds.

To prove (6), use Theorem 3.2, and observe that  $P_X \lesssim c_X$  by Theorem 2.5. To obtain (5), observe that  $P_X \lesssim c_X^3$ , by Theorem 2.4, and the condition of the eigenvalue estimate (7) is satisfied when  $X$  is a Cayley graph (indeed, for any vertex-transitive graph).  $\square$

### 3.1 Aside: A Reverse Poincaré Inequality for graphs

In the approaches of Colding and Minicozzi [8] and Kleiner [13], one also needs a “reverse Poincaré inequality” to control harmonic function on balls, while in the preceding proof we only need control of an eigenfunction on the entire graph (for which we could use (2)). We observe the following (perhaps known) version for eigenfunctions on graphs. An analogous statement holds in setting (M).

**Theorem 3.5.** *Suppose we are in the graph setting (G). Let  $\varphi : X \rightarrow \mathbb{R}$  be an eigenfunction of the Laplace operator with eigenvalue  $\lambda$ . Then,*

$$\int_{B(R)} |\nabla \varphi|^2 d\mu \leq \left( \frac{128}{dR^2} + 2\lambda \right) \int_{B(2R)} \varphi^2 d\mu.$$

The proof is based on the following lemma.

**Lemma 3.6.** Let  $\varphi : X \rightarrow \mathbb{R}$  be an eigenfunction of the Laplace operator with eigenvalue  $\lambda$ . Let  $u : X \rightarrow \mathbb{R}$  be a non-negative function that vanishes off  $B(R-1)$ , then

$$\int_{B(R)} u^2 |\nabla \varphi|^2 d\mu \leq \frac{128}{d} \int_{B(R)} \varphi^2 |\nabla u|^2 d\mu + 2\lambda \|u\|_\infty^2 \int_{B(R)} \varphi^2 d\mu.$$

*Proof.* Denote  $S = 2d \int_{B(R)} u |\nabla \varphi|^2 d\mu$ . Assume  $u$  vanishes off  $B(R-1)$ . We have,

$$\begin{aligned} S &= \int_{B(R)} \sum_{y \sim x} u(x)^2 |\varphi(x) - \varphi(y)|^2 d\mu(x) \\ &= \int_{B(R)} \sum_{y \sim x} u(x)^2 (\varphi(x)^2 + \varphi(y)^2 - 2\varphi(x)\varphi(y)) d\mu(x) \\ &= \int_{B(R)} \sum_{y \sim x} u(x)^2 \varphi(x)^2 d\mu(x) + \int_{B(R)} \sum_{y \sim x} u(x)^2 \varphi(y)^2 d\mu(x) - 2 \int_{B(R)} \sum_{y \sim x} u(x)^2 \varphi(x)\varphi(y) d\mu(x) \\ &= \int_{B(R)} \sum_{y \sim x} u(x)^2 \varphi(x)^2 d\mu(x) + \int_{B(R)} \sum_{y \sim x} u(y)^2 \varphi(x)^2 d\mu(x) - 2 \int_{B(R)} \sum_{y \sim x} u(x)^2 \varphi(x)\varphi(y) d\mu(x) \\ &= 2 \int_{B(R)} \sum_{y \sim x} u(x)^2 \varphi(x) (\varphi(x) - \varphi(y)) d\mu(x) + \int_{B(R)} \sum_{y \sim x} (u(y)^2 - u(x)^2) \varphi(x)^2 d\mu(x). \end{aligned}$$

Here we used that  $u(x)^2 \varphi(x)^2 = u(y)^2 \varphi(x)^2 = 0$  when  $y \notin B(R)$ , since  $u$  vanishes off  $B(R-1)$ . First, let us bound the first term.

$$\begin{aligned} \int_{B(R)} \sum_{y \sim x} u(x)^2 \varphi(x) (\varphi(x) - \varphi(y)) d\mu(x) &= \int_{B(R)} u(x)^2 \varphi(x) \left( \sum_{y \sim x} \varphi(x) - \varphi(y) \right) d\mu(x) \\ &= d \int_{B(R)} u^2 \varphi \Delta \varphi d\mu \\ &= d \int_{B(R)} u^2 \lambda \varphi^2 d\mu \\ &\leq d\lambda \|u\|_\infty^2 \int_{B(R)} \varphi^2 d\mu. \end{aligned}$$

Now we bound the second term.

$$\begin{aligned}
& \int_{B(R)} \sum_{y \sim x} (u(y)^2 - u(x)^2) \varphi(x)^2 d\mu(x) \\
& \leq 2 \int_{B(R)} \sum_{y \sim x} (u(y)^2 - u(x)^2) (\varphi(x)^2 - \varphi(y)^2) \\
& = 2 \left( \int_{B(R)} \sum_{y \sim x} |u(x) + u(y)|^2 |\varphi(x) - \varphi(y)|^2 d\mu(x) \right)^{1/2} \\
& \quad \times \left( \int_{B(R)} \sum_{y \sim x} |u(x) - u(y)|^2 |\varphi(x) + \varphi(y)|^2 d\mu(x) \right)^{1/2} \\
& \leq 8 \left( \int_{B(R)} \sum_{y \sim x} u(x)^2 |\varphi(x) - \varphi(y)|^2 d\mu(x) \right)^{1/2} \left( \int_{B(R)} \sum_{y \sim x} |u(x) - u(y)|^2 |\varphi(x)|^2 d\mu(x) \right)^{1/2} \\
& = 8S^{1/2} \left( \int_{B(R)} |\nabla u|^2 \varphi^2 d\mu \right)^{1/2}.
\end{aligned}$$

Combining these bounds we get,

$$S \leq 2d\lambda \|u\|_\infty^2 \int_{B(R)} \varphi^2 d\mu + 8S^{1/2} \left( \int_{B(R)} |\nabla u|^2 \varphi^2 d\mu \right)^{1/2}.$$

Therefore, either

$$S \leq 4d\lambda \|u\|_\infty^2 \int_{B(R)} \varphi^2 d\mu$$

and then we are done, or

$$S \leq 16S^{1/2} \left( \int_{B(R)} |\nabla u|^2 \varphi^2 d\mu \right)^{1/2}.$$

Then  $S \leq 256 \int_{B(R)} |\nabla u|^2 \varphi^2 d\mu$ . □

*Proof of Theorem 3.5.* The theorem follows from Lemma 3.6, if we choose

$$u(x) = \begin{cases} 1, & \text{if } x \in B(R) \\ 1 - d(x, B(R))/R, & \text{if } x \in B(2R) \setminus B(R). \end{cases}$$

□

### 3.2 Eigenvalue bounds

We now proceed to prove the eigenvalue bounds of Theorem 3.3. The following lemma is essentially proved in [11]; a similar statement with worse quantitative dependence can be deduced from [2].

**Lemma 3.7.** *There exists a constant  $A \geq 1$  such that the following holds. Let  $(X, \text{dist}, \mu)$  be any compact metric-measure space, where  $\mu$  satisfies the doubling condition with constant  $c_X$ . Then for any  $\tau > 0$ , there exists a finite partition  $P$  of  $X$  into  $\mu$ -measurable subsets such that the following holds. If  $S \in P$ , then  $\text{diam}(S) \leq \tau$ . Furthermore, if we use  $P(x)$  denote the set  $P(x) \in P$  which contains  $x \in X$ , then*

$$\mu \left( \left\{ x \in X : \text{dist}(x, X \setminus P(x)) \geq \frac{\tau}{A(1 + \log(c_X))} \right\} \right) \geq \frac{1}{2}. \quad (8)$$

*Proof.* We will first define a *random* partition of  $X$  as follows. Let  $N = \{x_1, x_2, \dots, x_M\}$  be a  $\tau/4$ -net in  $X$ , and choose a *uniformly random* bijection  $\pi : [M] \rightarrow [M]$ . Also let  $\alpha \in [\frac{1}{4}, \frac{1}{2}]$  be chosen uniformly at random, and inductively define

$$S_i = B(x_{\pi(i)}, \alpha\tau) \setminus \bigcup_{j=1}^{i-1} S_j.$$

It is clear that  $P = S_1 \cup S_2 \cup \dots \cup S_M$  forms a partition of  $X$  (note that some of the sets may be empty), and  $\text{diam}(S_i) \leq \tau$  for each  $i$ . Note that the distribution of  $P$  is independent of the measure  $\mu$ .

**Claim 3.8.** *For some  $A \geq 1$ , and every  $x \in X$ ,*

$$\Pr_P \left[ \text{dist}(x, X \setminus P(x)) \geq \frac{\tau}{A(1 + \log(c_X))} \right] \geq \frac{1}{2}. \quad (9)$$

By averaging, the claim implies that (8) holds for some partition  $P$  of the required form. For the sake of completeness, we include here a simple proof of Claim 3.8, which essentially follows from [5].

*Proof of Claim 3.8.* Fix a point  $x \in X$  and some value  $t \leq \tau/8$ . Observe that, by Fact 2.2, we have  $m = |N \cap B(x, \tau)| \leq c_X^{O(1)}$ . Order the points of  $N \cap B(x, \tau)$  in increasing distance from  $x$ :  $w_1, w_2, \dots, w_m$ . Let  $I_k = [d(x, w_k) - t, d(x, w_k) + t]$  and write  $\mathcal{E}_k$  for the event that  $\alpha\tau \leq d(x, w_k) + t$  and  $w_k$  is the minimal element according to  $\pi$  for which  $\alpha\tau \geq d(x, w_k) - t$ . It is straightforward to check that the event  $\{d(x, X \setminus P(x)) \leq t\}$  is contained in the event  $\bigcup_{k=1}^m \mathcal{E}_k$ . Therefore,

$$\begin{aligned} \Pr [d(x, X \setminus P(x)) \leq t] &\leq \sum_{k=1}^m \Pr[\mathcal{E}_k] = \sum_{k=1}^m \Pr[\alpha\tau \in I_k] \cdot \Pr[\mathcal{E}_k \mid \alpha\tau \in I_k] \\ &\leq \sum_{k=1}^m \frac{2t}{\tau/4} \frac{1}{k} \leq \frac{8t}{\tau} (1 + \log m), \end{aligned} \quad (10)$$

where we have used the fact that

$$\Pr[\mathcal{E}_k \mid \alpha\tau \in I_k] \leq \Pr[\min\{\pi(i) : i = 1, 2, \dots, k\} = \pi(k)] = 1/k.$$

Thus choosing  $t \approx \frac{\tau}{1 + \log(c_X)}$  in (10) yields the desired bound (9). □

□

The next simple lemma shows that on a “coarsely path-connected” space, a doubling measure cannot be concentrated on very small balls.

**Lemma 3.9.** *Let  $(X, \text{dist}, \mu)$  satisfy (G) or (M). Then for any  $x \in X$  and  $10 \leq R \leq D$ , we have*

$$\left(1 - \frac{1}{2c_X}\right) \mu(B(x, R)) \geq \mu(B(x, R/10)).$$

*Proof.* Let  $\delta = 1/(2c_X)$ . Suppose there is an  $x \in X$  with  $\mu(B(x, R/10)) \geq (1 - \delta)\mu(B(x, R))$ , and  $10 \leq R \leq D$ . We may assume that  $\mu(B(x, R)) = 1$ . In both settings (G) and (M), there exists a  $y \in X$  such that  $3R/5 \geq \text{dist}(x, y) \geq R/2$ . Let  $r = 3R/8$  so that  $B(y, 2r) \supseteq B(x, R/10)$  but  $B(y, r) \subseteq B(x, R) \setminus B(x, R/10)$ .

In this case,  $\mu(B(y, r)) \leq \mu(B(x, R)) - \mu(B(x, R/10)) \leq \delta$ , and

$$\mu(B(y, 2r)) = \mu(B(y, 3R/4)) > \mu(B(x, R/10)) \geq 1 - \delta \geq \frac{1 - \delta}{\delta} \mu(B(y, r)).$$

Since  $(1 - \delta)/\delta \geq c_X$ , this violates the doubling assumption, yielding a contradiction.  $\square$

**Corollary 3.10.** *Let  $(X, \text{dist}, \mu)$  satisfy (G) or (M). Then for any  $x \in X$  and any  $\varepsilon > 0$ , we have*

$$\mu(B(x, \varepsilon \text{diam}(X))) \leq 1 + \mu(X) \left(1 - \frac{1}{2c_X}\right)^{O(\log(\varepsilon^{-1}))}.$$

Under a symmetry assumption, there is an obvious improvement.

**Lemma 3.11.** *Let  $(X, \text{dist}, \mu)$  satisfy (G) or (M). If, for every  $x, y \in X$  and  $R \geq 0$ , we have  $\mu(B(x, R)) = \mu(B(y, R))$ , then for every  $\varepsilon > 0$ ,*

$$\mu(B(x, \varepsilon \text{diam}(X))) \lesssim 1 + \varepsilon \mu(X).$$

*Proof.* Fix  $x$  and  $y$  with  $\text{dist}(x, y) = \text{diam}(X)$ , and connect  $x$  and  $y$  by a geodesic  $\gamma$ . Let  $N \subseteq \gamma$  be a maximal  $(3\varepsilon \text{diam}(X))$ -separated set, so that  $|N| \gtrsim 1/\varepsilon$ . Then the balls  $\{B(u, \varepsilon \text{diam}(X))\}_{u \in N}$  are disjoint, and each of equal measure, implying the claim.  $\square$

We now prove Theorem 3.3, yielding upper bounds on the eigenvalues of  $\Delta$ .

*Proof of Theorem 3.3.* Use Corollary 3.10 to choose

$$\tau \geq \frac{\text{diam}(X)}{e^{O(c_X \log(k))}} \tag{11}$$

so that for every  $x \in X$ ,

$$\mu(B(x, 2\tau)) \leq \frac{\mu(X)}{8k}. \tag{12}$$

Let  $P$  be the partition guaranteed by Lemma 3.7 with parameter  $\tau$ . Since every  $S \in P$  satisfies  $\text{diam}(S) \leq \tau$ , (12) implies that  $\mu(S) \leq \mu(X)/(8k)$ . Call a set  $S \subseteq X$  *good* if it satisfies

$$\mu\left(\left\{x \in S : \text{dist}(x, X \setminus S) \geq \frac{\tau}{A(1 + \log(c_X))}\right\}\right) \geq \frac{1}{4}\mu(S). \tag{13}$$

By averaging, at least  $1/4$  of the measure is concentrated on good sets  $S \in P$ .

In particular, since every  $S \in P$  satisfies  $\mu(S) \leq \mu(X)/(8k)$ , from the good sets  $S \in P$ , we can form (by taking unions of small sets) disjoint sets  $S_1, S_2, \dots, S_k$  such that each  $S_i$  is good and satisfies

$$\frac{\mu(X)}{8k} \leq \mu(S_i) \leq \frac{\mu(X)}{4k}. \quad (14)$$

Now define  $f_i : X \rightarrow \mathbb{R}$  by  $f_i(x) = \text{dist}(x, X \setminus S_i)$ . Clearly the  $f_i$ 's have disjoint support. Furthermore, each  $f_i$  is 1-Lipschitz, hence  $\int |\nabla f_i|^2 d\mu \leq \mu(X)$ . Finally, since each set  $S_i$  is good and satisfies (14), we have

$$\int (f_i - \bar{f}_i)^2 d\mu \gtrsim \frac{\tau^2}{(\log c_X)^2} \frac{\mu(X)}{k}.$$

Using (3), this implies that

$$\lambda_k \leq \frac{k^{O(c_X)}}{\text{diam}(X)^2}.$$

Observe that we can obtain a better bound

$$\lambda_2 \leq \frac{c_X^{O(1)}}{\text{diam}(X)^2}$$

as follows. In this case, we only need one test function. Choose  $\tau = \text{diam}(X)/20$  above, and use Lemma 3.9 to form a good set  $S_1$  which satisfies

$$\frac{\mu(X)}{2c_X} \leq \mu(S_1) \leq \left(1 - \frac{1}{2c_X}\right) \mu(X),$$

then define  $f_1(x) = \text{dist}(x, X \setminus S_1)$ .

Finally, to prove (7), note that under the measure symmetry assumption, we can employ Lemma 3.11 to choose  $\tau \geq \frac{\text{diam}(X)}{O(k)}$  in (11). The rest of the proof proceeds exactly as before.  $\square$

## 4 Applications to finite groups

We now give some applications of Theorem 3.1 to finite groups.

**Theorem 4.1.** *Let  $G$  be a finite group with symmetric generating set  $S$ . Let  $c_G = \max_{R>0} \frac{|B(2R)|}{|B(R)|}$  be the doubling constant of the Cayley graph  $\text{Cay}(G; S)$ . Then there exists a finite-dimensional representation  $\rho_W : G \rightarrow GL(W)$  such that*

1.  $\dim W \leq \exp(O(\log c_G)^2)$ ,
2.  $|\rho_W(G)| \gtrsim |G|^{1/\log_2 c_G} / c_G^{O(1)}$ .

*Proof.* Without loss of generality, we assume that  $c_G \geq 2$  throughout. Recall that  $d = |S|$ .

Consider the action of  $G$  on  $L^2(G)$  via  $[\rho(g)f](x) = f(g^{-1}x)$ . Note that this action commutes with the Laplacian,

$$[\Delta \rho(g)f](x) = [\Delta f](g^{-1}x) = f(g^{-1}x) - \frac{1}{d} \sum_{s \in S} f(g^{-1}xs) = [\Delta f](g^{-1}x) = [\rho(g)\Delta f](x).$$

Therefore, every eigenspace of the Laplacian is invariant under the action of  $G$ . Now let  $W \equiv W_2$  and  $\lambda_2$  be the second eigenspace and eigenvalue, respectively, of the Laplacian on  $\text{Cay}(G; S)$ . Let  $\rho_W$  be the restriction of  $\rho$  to  $W$ . First, by Theorem 3.1(5),  $\dim W \leq \exp(O(\log c_G)^2)$ .

Now we need to prove the lower bound on  $|\rho_W(G)|$ . By Theorem 3.3, we have

$$\lambda_2 \lesssim \frac{c_G^{O(1)}}{\text{diam}(\text{Cay}(G; S))^2}. \quad (15)$$

Consider  $H = \ker \rho_W$ , the set of elements which act trivially on  $W$ .  $H$  is a normal subgroup of  $G$  and  $\rho_W(G) \cong G/H$ . Let  $f$  be an arbitrary non-zero function in  $W_2$ . Note that  $f$  is constant on every coset  $Hg$  since the value of  $f(hg) = [\rho(h^{-1})f](g) = f(g)$  does not depend on  $h \in H$ . Define  $\hat{f} : G/H \rightarrow \mathbb{R}$  by  $\hat{f}(Hg) = f(g)$ . Observe that  $\hat{f}$  is a non-constant eigenfunction of the Laplacian on the quotient graph  $\text{Cay}(G/H; S)$  with eigenvalue  $\lambda_2$ ,

$$\Delta \hat{f}(Hg) = \hat{f}(Hg) - \frac{1}{d} \sum_{s \in S} \hat{f}(Hgs) = f(g) - \frac{1}{d} \sum_{s \in S} f(gs) = \Delta f(g) = \lambda_2 f(g) = \lambda_2 \hat{f}(Hg).$$

Let  $\lambda_2(G/H)$  denote the second eigenvalue of the Laplacian on  $\text{Cay}(G/H; S)$ . Since  $\lambda_2$  is a non-zero eigenvalue of the Laplacian on  $\text{Cay}(G/H; S)$ , we have  $\lambda_2(G/H) \leq \lambda_2$ . However, by the discrete Cheeger inequality [1],

$$\lambda_2(G/H) \geq \frac{h(\text{Cay}(G/H; S))^2}{2d^2} \quad (16)$$

where  $h(\text{Cay}(G/H; S))$  is the Cheeger constant of  $\text{Cay}(G/H; S)$ :

$$h(\text{Cay}(G/H; S)) \equiv \max_{U \subset G/H; |U| \leq |G/H|/2} \frac{E(U, (G/H) \setminus U)}{|U|} \geq \frac{1}{|G/H|/2},$$

here  $E(U, (G/H) \setminus U)$  denotes the set of edges between  $U$  and  $(G/H) \setminus U$  in  $\text{Cay}(G/H; S)$ , and the bound follows because  $\text{Cay}(G/H; S)$  is a connected graph.

We conclude that

$$\lambda_2 \geq \lambda_2(G/H) \geq \frac{(2/|G/H|)^2}{2d^2} = \frac{2}{(d|G/H|)^2}.$$

Combining this bound with (15), we get

$$|\rho_W(G)| = |G/H| \geq \sqrt{\frac{2}{d^2 \lambda_2}} \gtrsim \frac{\text{diam}(\text{Cay}(G; S))}{c_G^{O(1)}}.$$

The desired bound now follows using the fact that  $\text{diam}(\text{Cay}(G; S)) \geq |G|^{1/\log_2 c_G}$ .  $\square$

**Corollary 4.2.** *Under the assumptions of Theorem 4.1, there exists a normal subgroup  $N$  with  $[G : N] \leq \alpha$  such that  $N$  has  $\mathbb{Z}_M$  as a homomorphic image, where  $M \gtrsim |G|^\delta$  and  $\delta = \delta(c_G)$  and  $\alpha = \alpha(c_G)$  depend only on the doubling constant of  $G$ .*

*Proof.* Let  $\rho_W : G \rightarrow GL(W)$  be the representation guaranteed by Theorem 4.1, and put  $k = \dim W$ . Now,  $H = \rho_W(G)$  is a finite subgroup of  $GL(W)$ , hence by a theorem of Jordan (see [9, 36.13]),  $H$  contains a normal abelian subgroup  $A$  with  $[H : A] = O(k)^{k^2}$ . Since  $A$  is abelian, its members can be simultaneously diagonalized over  $\mathbb{C}$ ; it follows that  $A$  is a product of at most  $k$  cyclic groups, hence  $\mathbb{Z}_M \leq A$  for some  $M \geq |A|^{1/k}$ . Putting  $N = \rho^{-1}(A)$ , we see that  $[G : N] = [H : A] = O(k)^{k^2}$ , and  $N$  maps homomorphically onto  $\mathbb{Z}_M$ .  $\square$

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