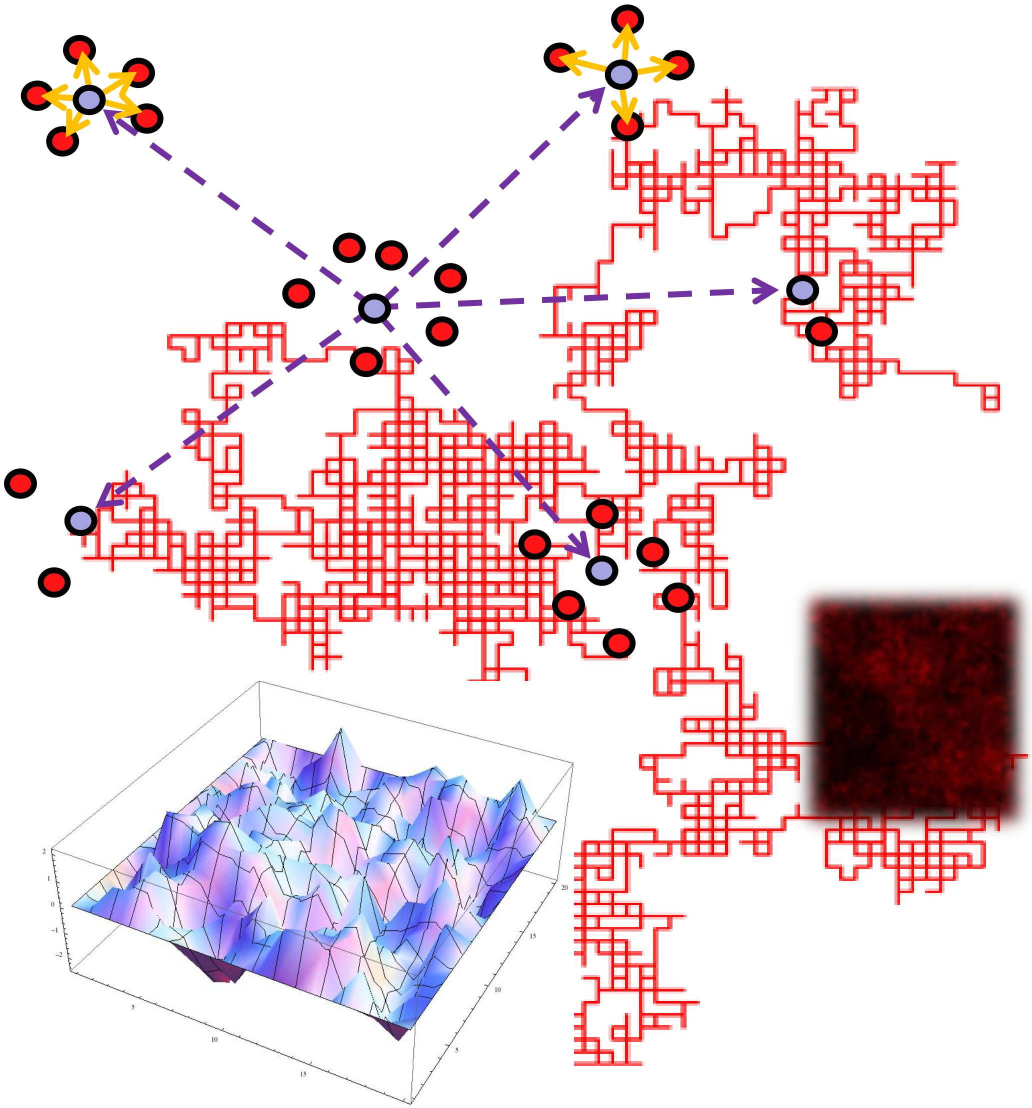
cover times, blanket times, and the GFF



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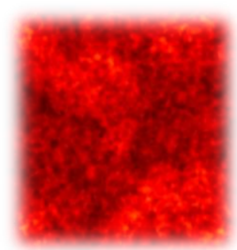
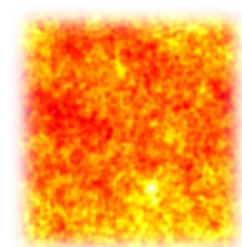
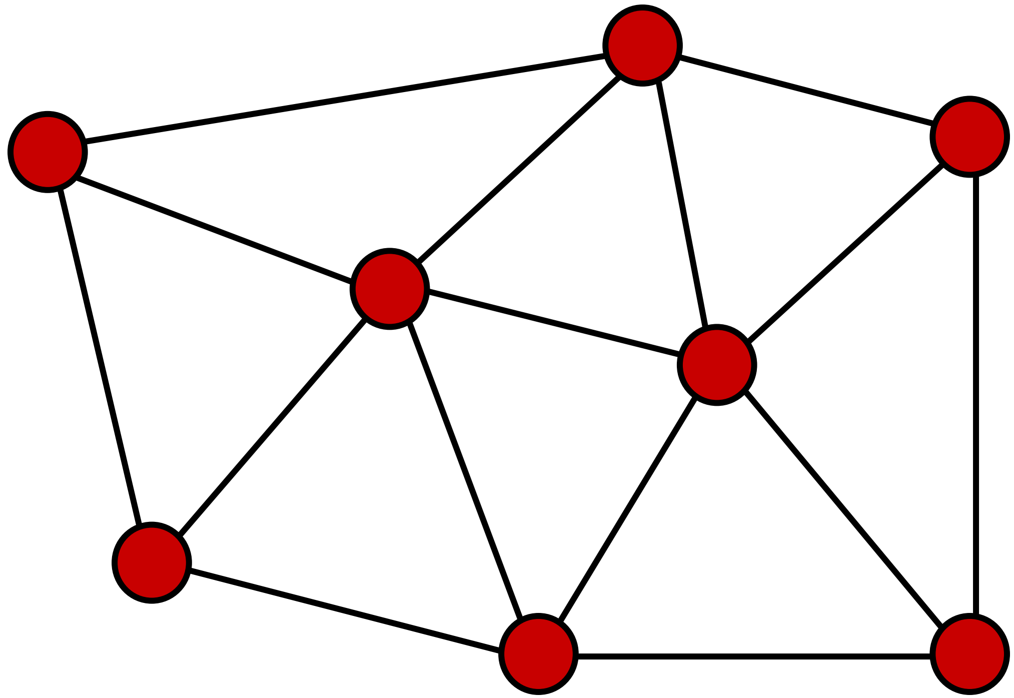




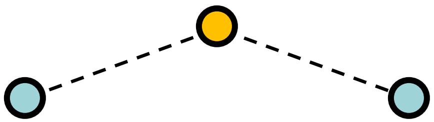
random walks on graphs



By putting conductances $\{c_{uv}\}$ on the edges of the graph, we can get any reversible Markov chain.

Hitting time: $H(u,v)$ = expected # of steps to hit v starting at u

Commute time: (metric) $\kappa(u,v) = H(u,v) + H(v,u)$

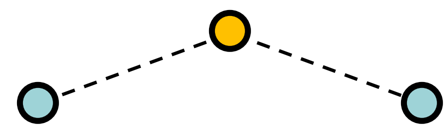


Cover time: $t_{cov}(G)$ = expected time to hit all vertices of G, starting from the **worst** vertex

hitting and covering

Hitting time: $H(u,v)$ = expected # of steps to hit v starting at u

Commute time: $\kappa(u,v) = H(u,v) + H(v,u)$
(metric)



Cover time: $t_{cov}(G)$ = expected time to hit all vertices of G, starting from the **worst** vertex

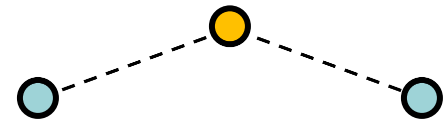
orders of magnitude of some cover times

path	n^2	
complete graph	$n \log n$	[coupon collecting]
expander	$n \log n$	[Broder-Karlin 88]
2-dimensional grid	$n (\log n)^2$	[Aldous 89, Zuckerman 90]
3-dimensional grid	$n \log n$	[Aldous 89, Zuckerman 90]
complete d-ary tree	$n (\log n)^2/\log d$	[Zuckerman 90]

hitting and covering

Hitting time: $H(u,v)$ = expected # of steps to hit v starting at u

Commute time: $\kappa(u,v) = H(u,v) + H(v,u)$
(metric)



Cover time: $t_{cov}(G)$ = expected time to hit all vertices of G, starting from the **worst** vertex

general bounds (n = # vertices, m = # edges)

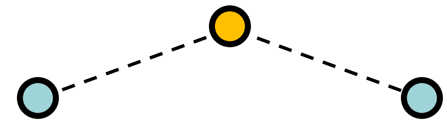
$$(1-o(1))\, n \log n \le t_{cov}(G) \le 2nm$$

[Feige'95, Matthews'88] [Alelinuas-Karp-Lipton-Lovasz-Rackoff'79]

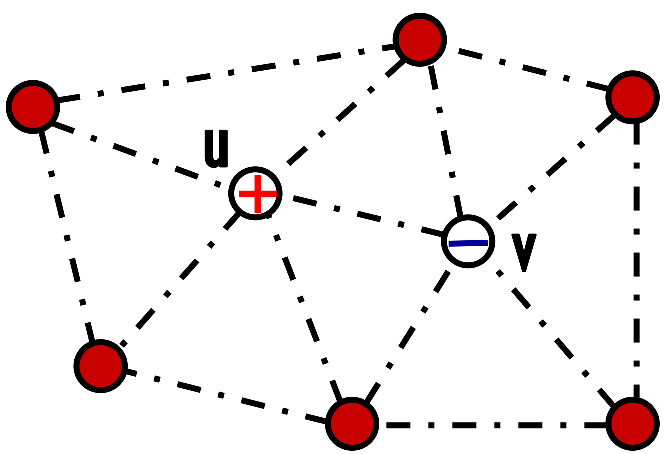
electrical resistance

Hitting time: $H(u,v)$ = expected # of steps to hit v starting at u

Commute time: $\kappa(u,v) = H(u,v) + H(v,u)$
(metric)



Cover time: $t_{cov}(G)$ = expected time to hit all vertices of G, starting from the **worst** vertex



$R_{\text{eff}}(u,v)$ = inverse of electrical current flowing from u to v

[Chandra-Raghavan-Ruzzo-Smolensky-Tiwari'89]:

If G has m edges, then for every pair u,v

$$\kappa(u,v) = 2m\, R_{\text{eff}}(u,v)$$

(endows κ with certain geometric properties)

computation

Hitting time: Easy to compute in deterministic poly time by solving system of linear equations

$$H(u,u) = 0$$
$$H(u,v) = 1 + \frac{1}{d_u}\sum_{w\sim u} H(w,v)$$

Cover time: Easy to compute in exponential time by a determinsitic algorithm and polynomial time by a randomized algorithm

Natural question: Does there exist a poly-time deterministically computable $O(1)$-approximation for general graphs?

[Aldous-Fill'94]

approximation in derministic poly-time

Clearly: $H_{\max} = \max_{u,v \in V} H(u,v) \cdot \ t_{\text{cov}}$

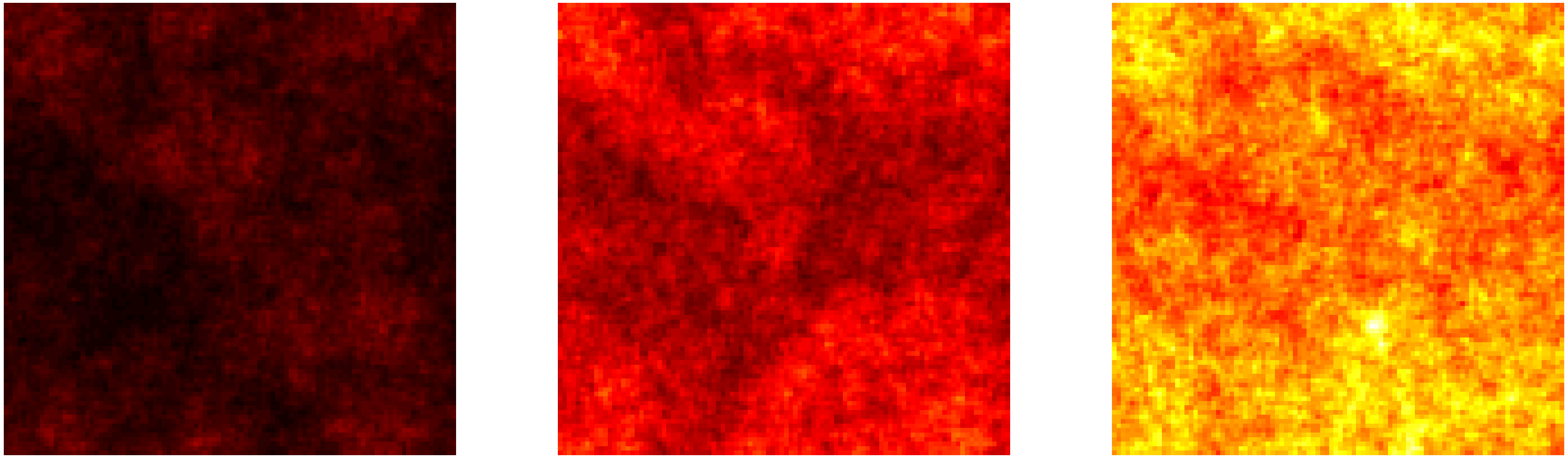
[Matthews'88] proved: $t_{\text{cov}} \cdot \ H_{\max}(1 + \log n)$

and $t_{\text{cov}} \geq \max_{S \subseteq V} \min_{u,v \in S} H(u,v)(\log |S| - 1)$

[Kahn-Kim-Lovasz-Vu'99] show that Matthews' lower bound gives an $O(\log \log n)^2$ approximation to t_{cov}

[Feige-Zeitouni'09] give a $(1+\epsilon)$-approximation for trees for every $\epsilon > 0$, using recursion.

blanket times



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Blanket times [Winkler-Zuckerman'96]:

The β**-blanket time** $t_{\text{blanket}}(G,\beta)$ is the expected first time **T** at which all the local times,

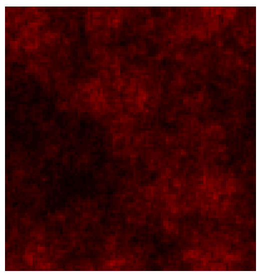
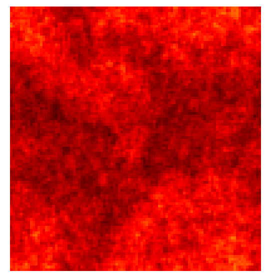
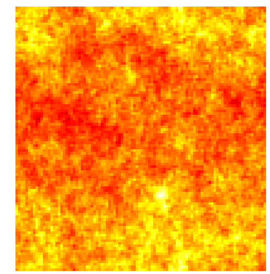
$$L_T^x = \frac{\#\text{ visits to } x}{\pi(x)}$$

are within a factor of β.

blanket time conjecture

Conjecture [Winkler-Zuckerman'96]:

For every graph G and $0 < \beta < 1$, $t_{\text{blanket}}(G,\beta) \asymp t_{\text{cov}}(G)$.

 $\asymp$  $\asymp$ 

Proved for many special cases.

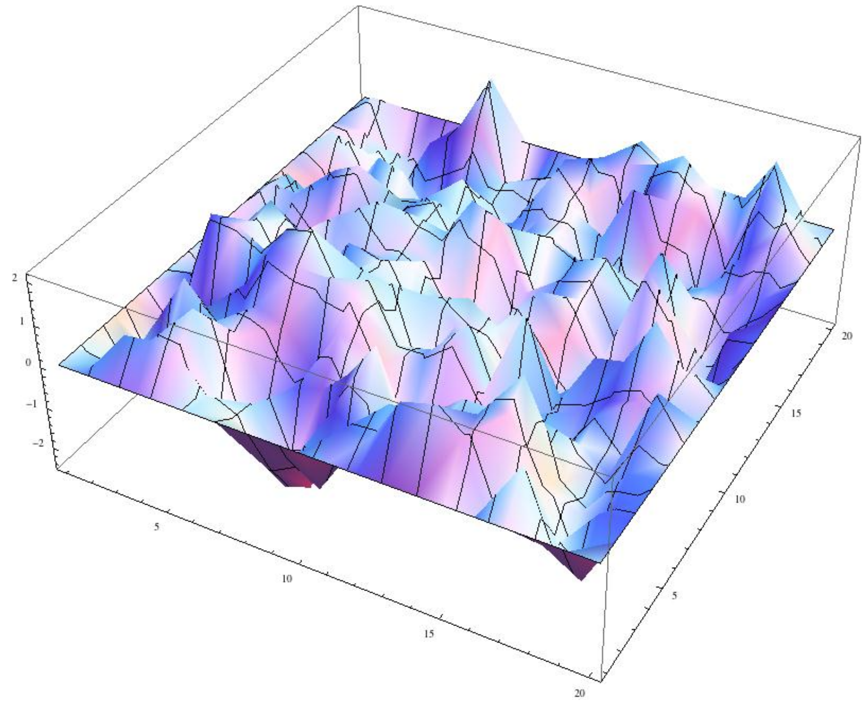
True up to **$(\log \log n)^2$** by [Kahn-Kim-Lovasz-Vu'99]

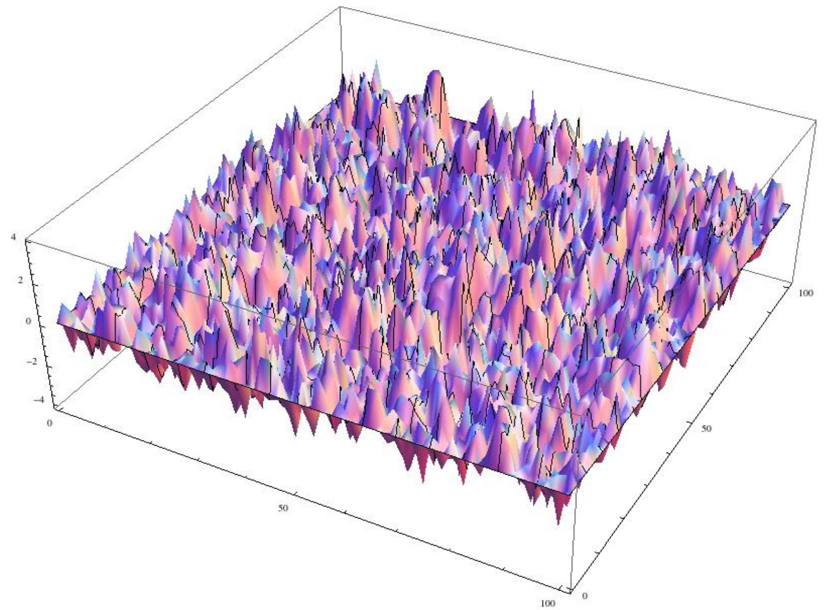
Gaussian free field on a graph

A centered Gaussian process $\{g_v\}_{v\in V}$ satisfying for all $u,v \in V$:

$$\mathbb{E}\,(g_u - g_v)^2 = R_{\mathsf{eff}}(u,v)$$

and $g_{v_0} = 0$ for some fixed $v_0 \in V$.





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and $g_{v_0} = 0$ for some fixed $v_0 \in V$.

Density proportional to

$$e^{-\frac{1}{2}\sum_{u\sim v}|g_u - g_v|^2}$$

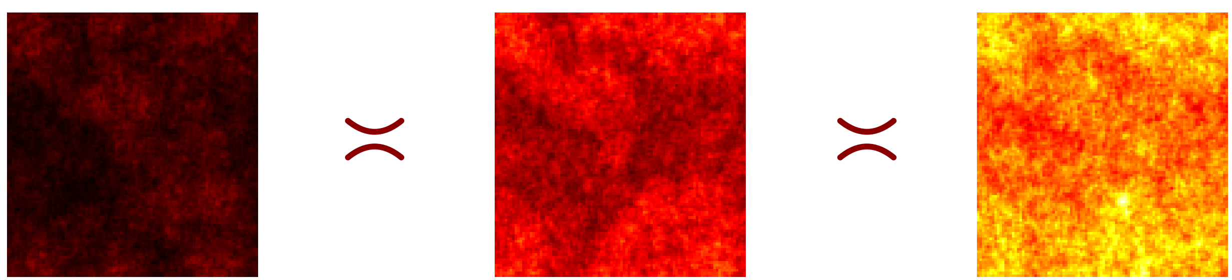
main result

For every graph $G=(V,E)$,

$$t_{\mathrm{cov}}(G) \asymp |E| \left(\mathbb{E} \max_{v \in V} g_v \right)^2 \asymp_\beta t_{\mathrm{blanket}}(G, \beta)$$

for every $0 < \beta < 1$, where $\{g_v\}_{v \in V}$ is the GFF on G.

Positively resolves the Winkler-Zuckerman blanket time conjectures.

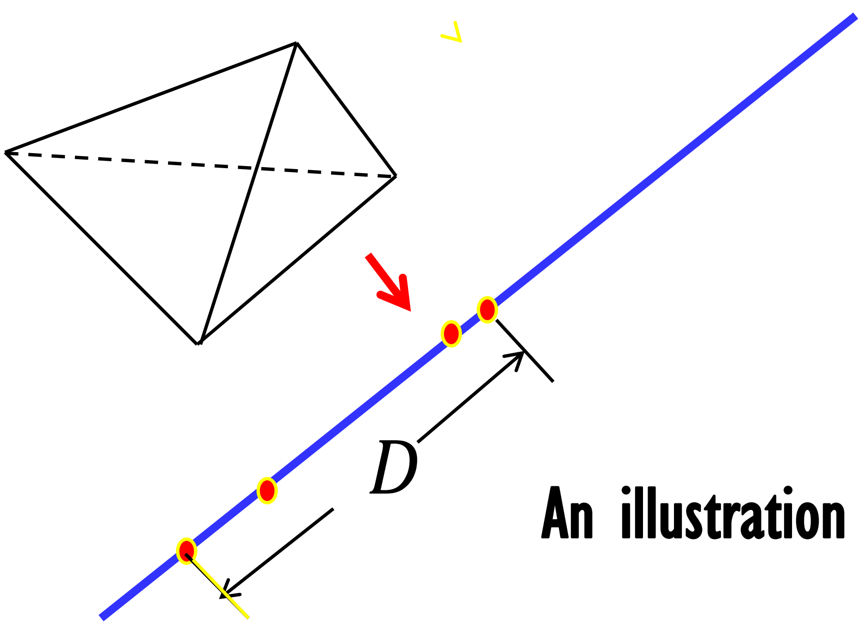


geometrical interpretation

$(G, \sqrt{R_{\text{eff}}})$ embedded in Euclidean space via GFF.

$$t_{\text{cov}}(G) \asymp n \cdot |E| (\mathbb{E}D)^2$$

where D is the diameter of the projection of this embedding onto a random line:



An illustration for the complete graph K_4

Gaussian processes

Consider a **Gaussian process** $\{X_u : u \in S\}$ with $\mathbb{E}\,(X_u)=0 \quad \forall u \in S$

Such a process comes with a natural metric

$$d(u,v) = \sqrt{\mathbb{E}\,(X_u - X_v)^2}$$

transforming (S, d) into a metric space.

PROBLEM: What is $\mathbb{E}\max\{\,X_u : u \in S\,\}$?

Majorizing measures theorem [Fernique-Talagrand]:

For every Gaussian process $\{X_u\}_{u \in S}$,

$$\mathbb{E} \sup_{u \in S} X_u \asymp \gamma_2(S, d)$$

where $d(u, v) = \sqrt{\mathbb{E}\,(X_u - X_v)^2}$.

γ_2 is a functional on metric spaces, given by an explicit formula which takes exponential time to calculate from the definition

main theorem, restated

THEOREM: For every graph $G=(V,E)$,

$$t_{\mathsf{cov}}(G) \asymp \left(\gamma_2(V, \sqrt{\kappa})\right)^2$$

where κ = commute time.

We also construct a deterministic poly-time algorithm to approximate γ_2 within a constant factor.

COROLLARY: There is a deterministic poly-time $O(1)$-approximation for t_{cov}

Positively answers the question of Aldous and Fill.

Majorizing measures theorem [Fernique-Talagrand]:

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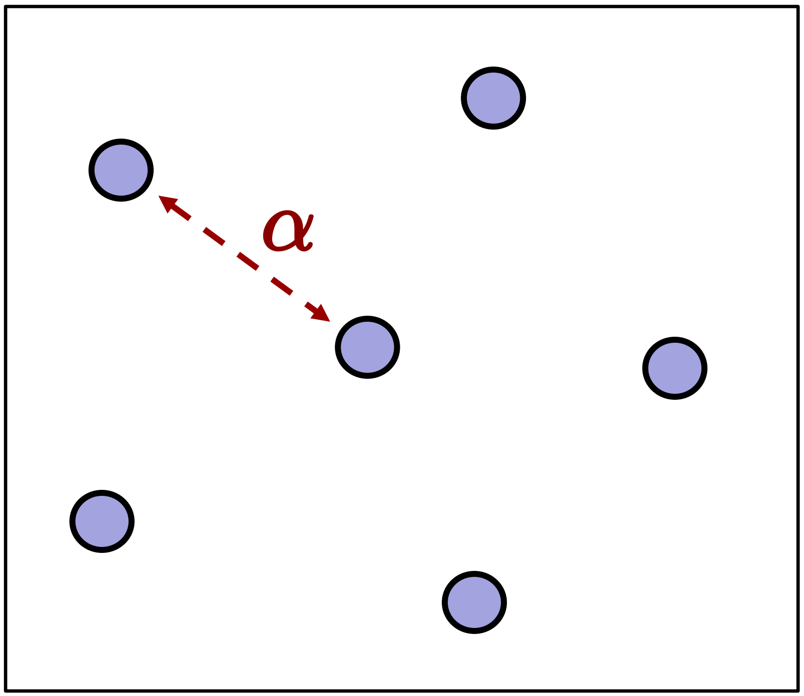
PROBLEM: What is $\mathbb{E}\max\{X_u : u \in S\}$?



If random variables are "independent," expect the union bound to be tight.

Expect **max** for k points is about

$$\alpha\sqrt{\log k}$$

Gaussian concentration:

$$\Pr(X_u - X_v > \lambda) \leq \exp\left(\frac{-\lambda^2}{2\,d(u,v)^2}\right)$$

Gaussian processes

Gaussian concentration:

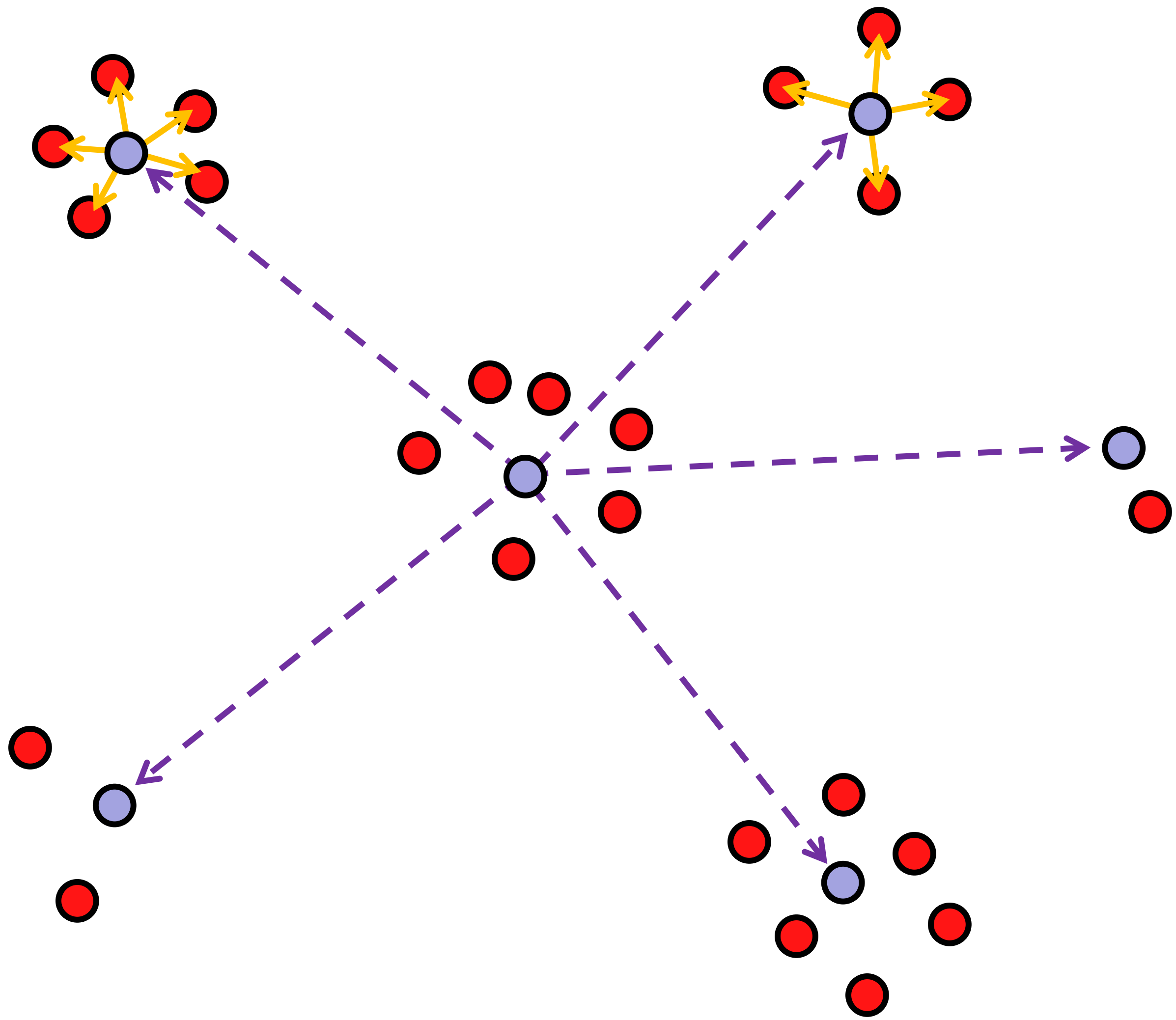
$$\Pr\left(X_u - X_v > \lambda\right) \leq \exp\left(\frac{-\lambda^2}{2\,d(u,v)^2}\right)$$

Sudakov minoration:

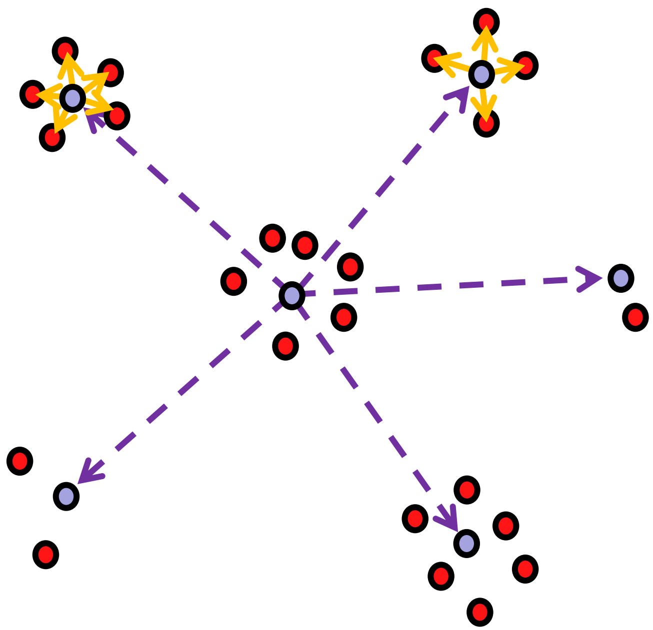
If $\{X_u\}_{u \in S}$ is a Gaussian process and $d(u,v) \geq \alpha$ for all $u \neq v$, then

$$\mathbb{E}\max\{X_u : u \in S\} \gtrsim \alpha\sqrt{\log |S|}$$

chaining



chaining



- Chaining with a symmetric tree gives Dudley's (1967) entropy bound

- Best possible tree upper bound yields γ_2

[Dudley'67]: $$\mathbb{E} \sup_{s \in S} X_s \lesssim \int_0^\infty \sqrt{\log N(S,d,\varepsilon)}\, d\varepsilon$$

where $N(S,d,\epsilon)$ is the minimal number of ϵ-balls needed to cover S

hints of a connection

Gaussian concentration:

$$\Pr\left(X_u - X_v > \lambda\right) \le \exp\left(\frac{-\lambda^2}{2\,d(u,v)^2}\right)$$

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Matthew's bound (1988):

If $S \subseteq V$ is a subset of nodes in G such that $\kappa(u,v) \geq \alpha$ for all $u \neq v \in S$, then

$$t_{\mathrm{cov}}(G) \geq \alpha \log |S|$$

hints of a connection

Gaussian concentration:

$$\Pr\left(X_u - X_v > \lambda\right) \le \exp\left(\frac{-\lambda^2}{2\,d(u,v)^2}\right)$$

KKLV'99 concentration:

For all nodes $u, v \in V, \alpha > 0$ and $\ell \ge 0$,

$$\mathbb{P}(L_T^u - L_T^v > \alpha) \le \exp\left(-\frac{\alpha^2}{4\ell R_{\text{eff}}(u,v)}\right)$$

$T = T(\ell) =$ global time.
L_T^v is local time at v and $L_T^u = \ell$.

hints of a connection

Dudley'67 entropy bound:

$$\mathbb{E} \sup_{s \in S} X_s \lesssim \int_0^\infty \sqrt{\log N(S, d, \varepsilon)}\, d\varepsilon$$

Barlow-Ding-Nachmias-Peres'09 analog for cover times

Dynkin isomorphism theory

Recall the *local time of* v *at time* t is given by

$$L_t^v = \frac{\#\text{ visits to } v}{\deg(v)}.$$

Ray-Knight (1960s):
Characterize the local times of Brownian motion.

Dynkin (1980):
General connection of Markov process to Gaussian fields.

The version we used is due to
Eisenbaum, Kaspi, Marcus, Rosen, and **Shi** (2000)

Generalized Ray-Knight theorem

For $v_0 \in V$, define $T(\ell) = \inf \{t: L_t^{v_0} \geq \ell\}$.

Let g be the GFF on G with $g_{v_0} = 0$. Then

$$\left\{L_T^x + \frac{1}{2} g_x^2 : x \in V\right\} \overset{law}{=} \left\{\frac{1}{2}\left(g_x - \sqrt{2\ell}\right)^2 : x \in V\right\}$$

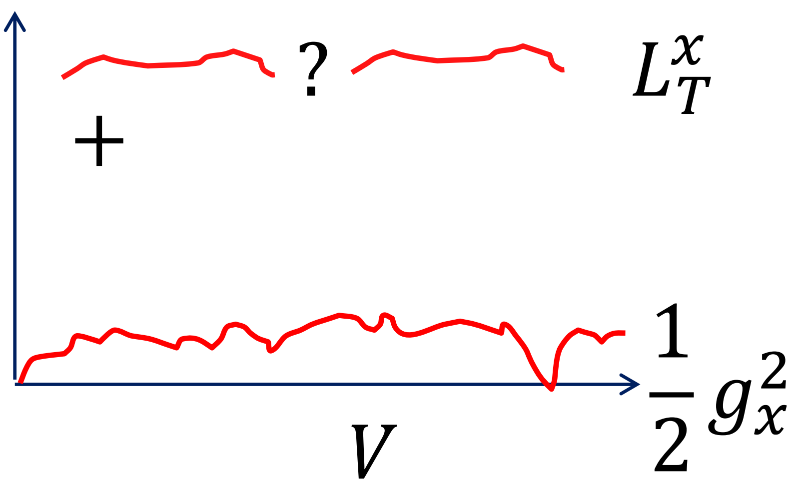
where $T = T(\ell)$.

blanket time vs. GFF

$$M = \max_x g_x \qquad \ell \gg (\mathbb{E}M)^2$$

$$T = T(\ell)$$

$$L_T^x + \frac{1}{2}g_x^2 \overset{law}{=} \ell - \sqrt{2\ell}\cdot g_x + \frac{1}{2}g_x^2$$



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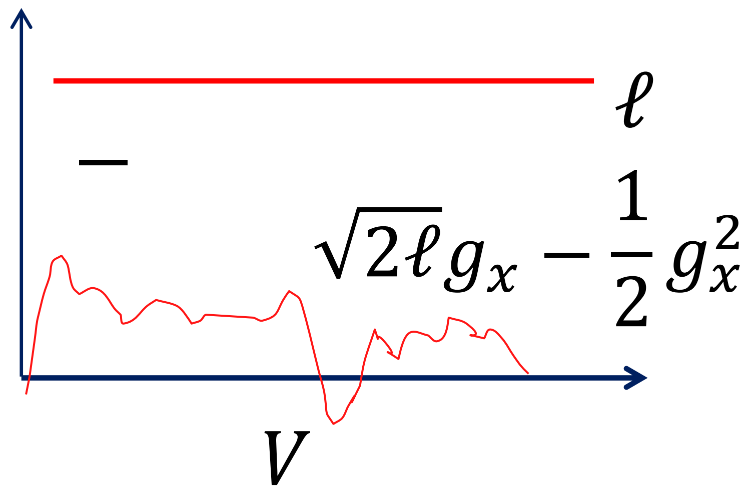


$$\left\{L_T^x + \frac{1}{2}\, g_x^2 : x \in V\right\} \overset{law}{=} \left\{\frac{1}{2}\left(g_x - \sqrt{2\ell}\right)^2 : x \in V\right\}$$

$T = T(\ell)$

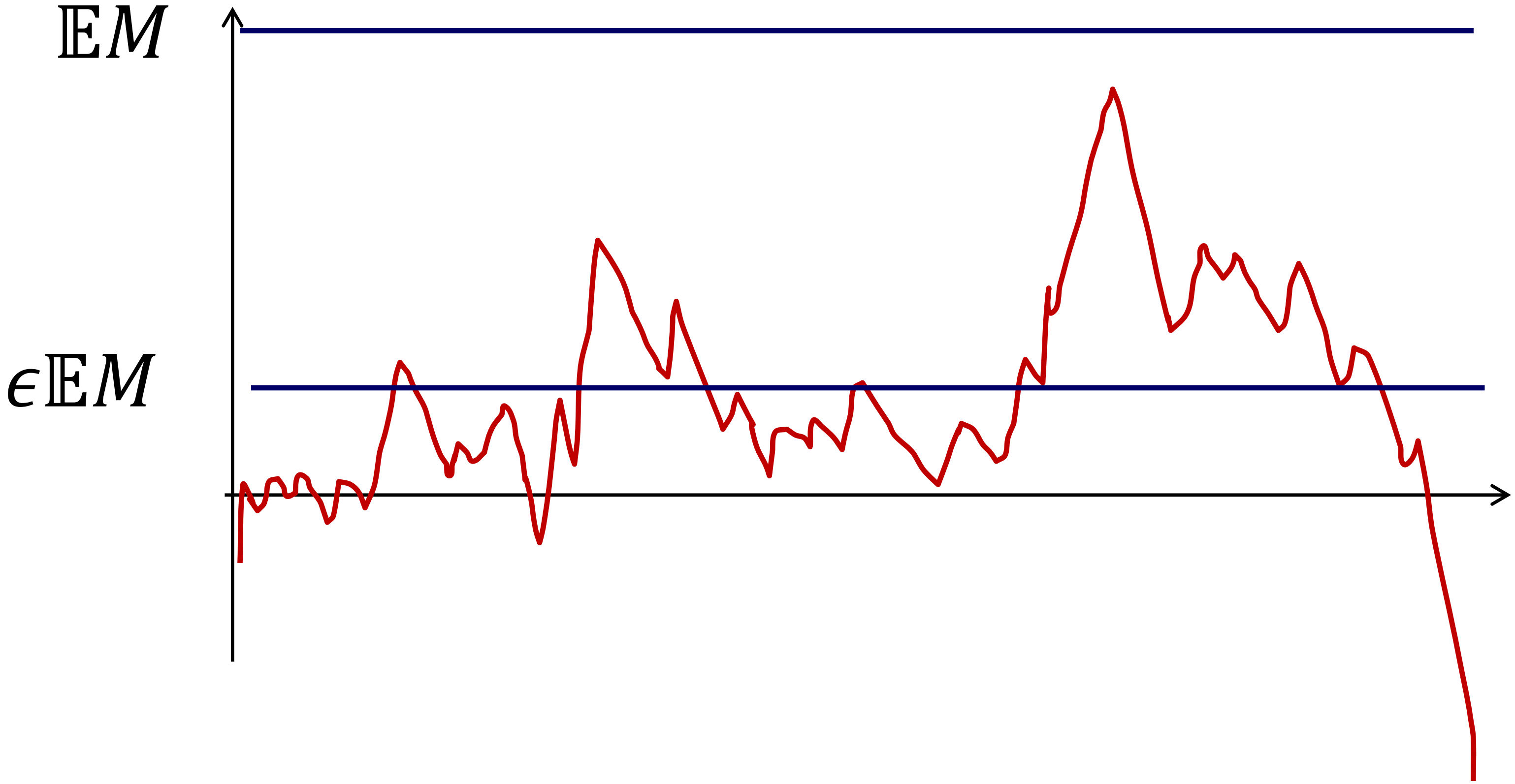
To lower bound cover time:

Need to show that for $T = \epsilon |E| (\mathbb{E}M)^2$,

with good chance $\exists x \in V : L_T^x = 0$.

$|\sqrt{2\ell} - g_x|$ small $\Rightarrow L_T^x$ small $\Rightarrow L_T^x = 0$, w.h.p.

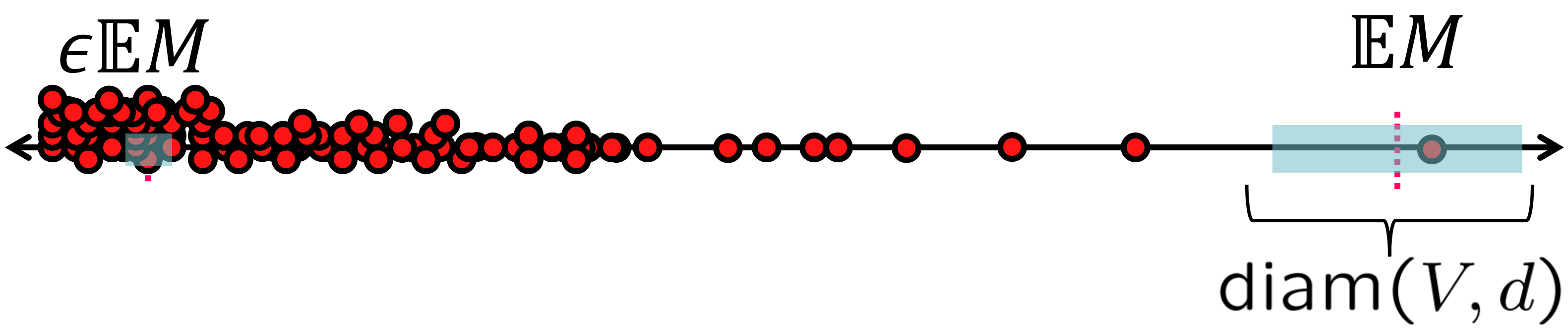
only one Everest, but...



a problem on Gaussian processes

Gaussian free field: $\{g_x\}_{x \in V}$

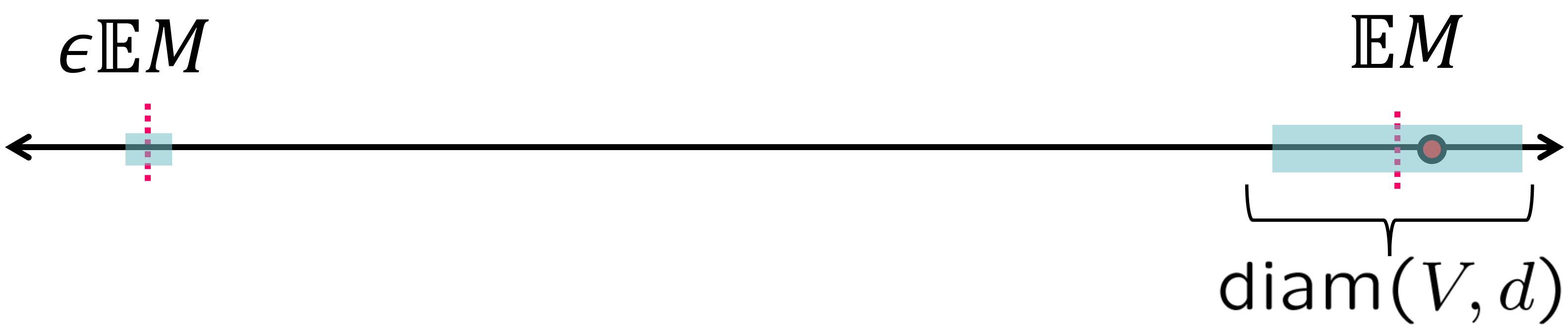
$$M = \max_{x \in V} g_x$$



a problem on Gaussian processes

Gaussian free field: $\{g_x\}_{x \in V}$

$$M = \max_{x \in V} g_x$$

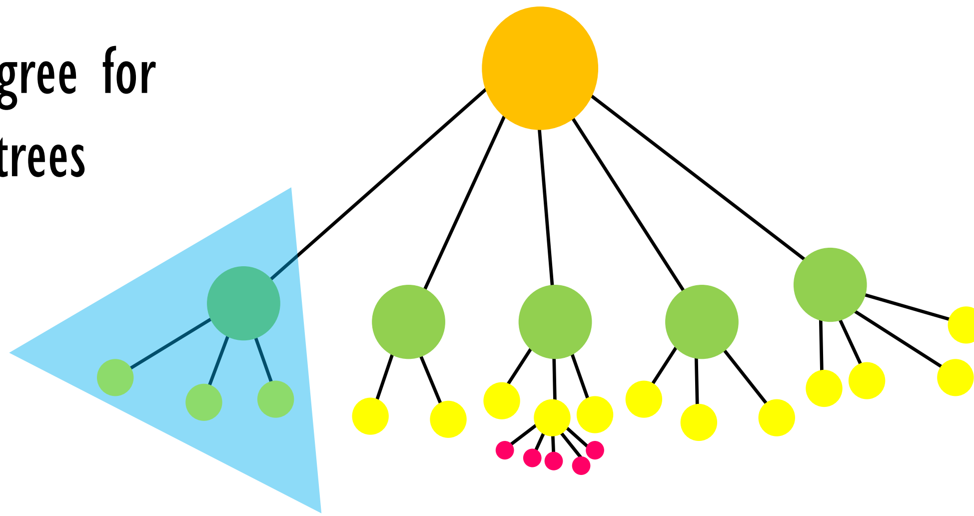


We need strong estimates on the size of this window.
(want to get a point there with probability at least 0.1)

Problem: Majorizing measures handles first moments, but we need second moment bounds.

percolation on trees and the GFF

First and second moments agree for percolation on balanced trees

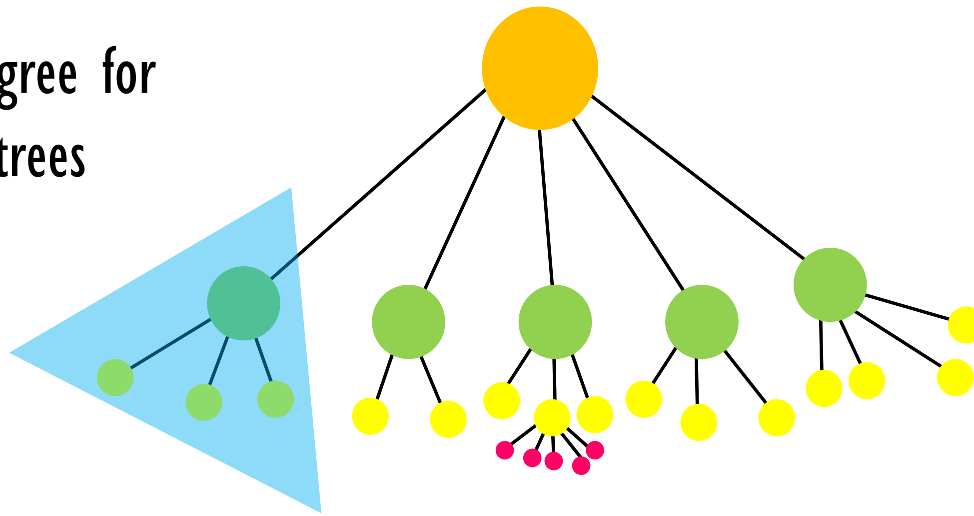


Problem: General Gaussian processes behaves nothing like percolation!

Resolution: Processes coming from the **Isomorphism Theorem** all arise from a GFF

percolation on trees and the GFF

First and second moments agree for percolation on balanced trees



For DGFFs, using electrical network theory, we show that it is possible to select a subtree of the MM tree and a delicate filtration of the probability space so that the Gaussian process can be coupled to a percolation process.

example: precise asymptotics in 2D

Bolthausen, Deuschel and Giacomin (2001):

For Gaussian free field g_x on $2D$ lattice of n vertices

$$\mathbb{E} \sup_x g_x \sim \sqrt{\frac{1}{2\pi}} \log n \,.$$

Dembo, Peres, Rosen, and Zeitouni (2004):

For 2D torus, $t_{\mathrm{cov}} \sim \frac{1}{\pi} n \,(\log n)^2$.

$$t_{\mathrm{cov}} \sim |E| (\mathbb{E} \sup_x g_x)^2$$

open questions

$$t_{\mathrm{cov}} \sim |E|(\mathbb{E} \sup_{x \in V} g_x)^2 \quad ?$$

Upper bound is true.

Lower bound holds for:

- complete graph
- complete d-ary tree
- discrete 2D torus

QUESTION: Is there a deterministic, polynomial-time $(1+\epsilon)$-approximation to the cover time for every $\epsilon > 0$?

QUESTION: Is the standard deviation of the time-to-cover bounded by the maximum hitting time?