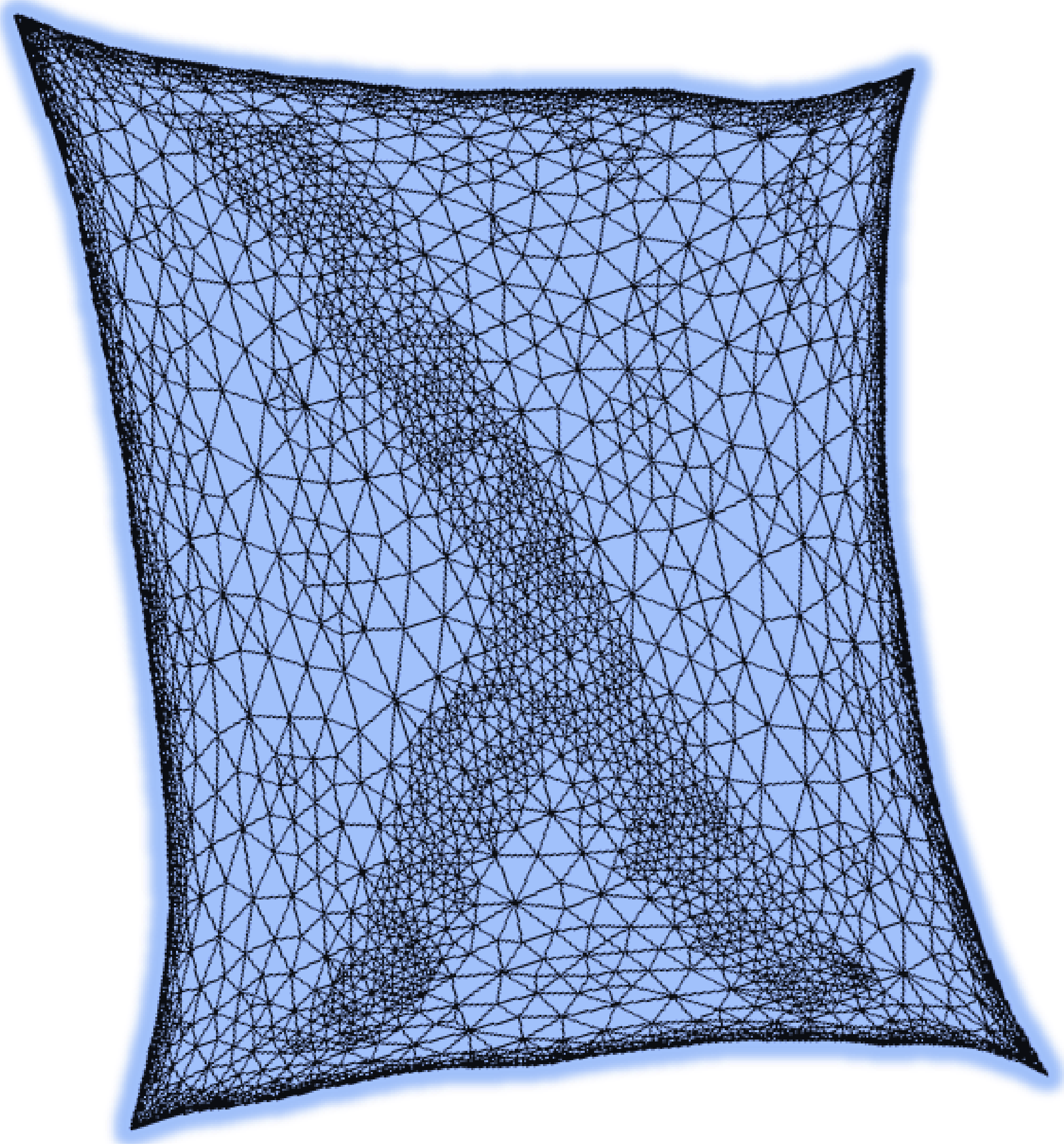


Real symmetric matrices have real eigenvalues and eigenvectors.

$$A_{ij} = A_{ji}$$

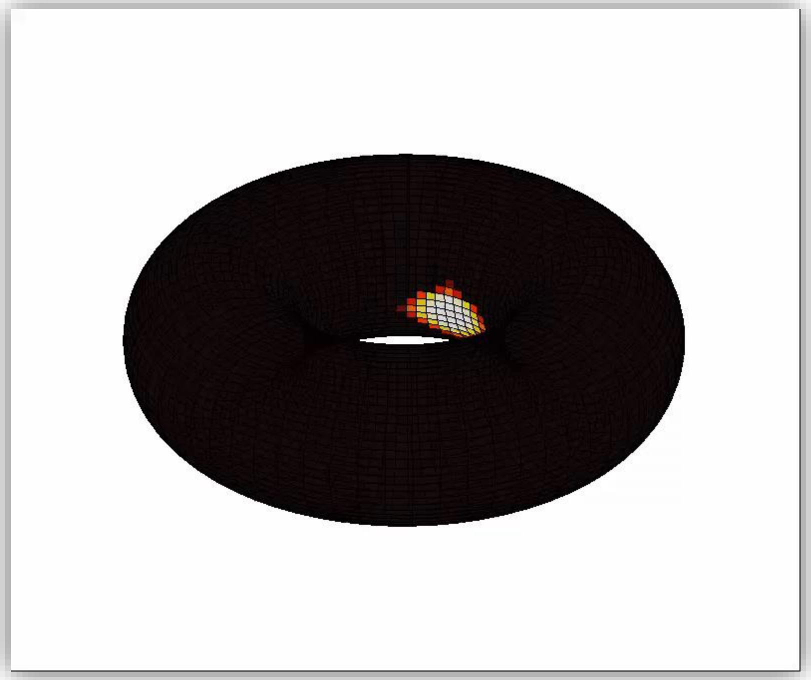
$$A = \begin{pmatrix} 2 & -1 & 3 \\ -1 & -2 & -1 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = (-1) \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

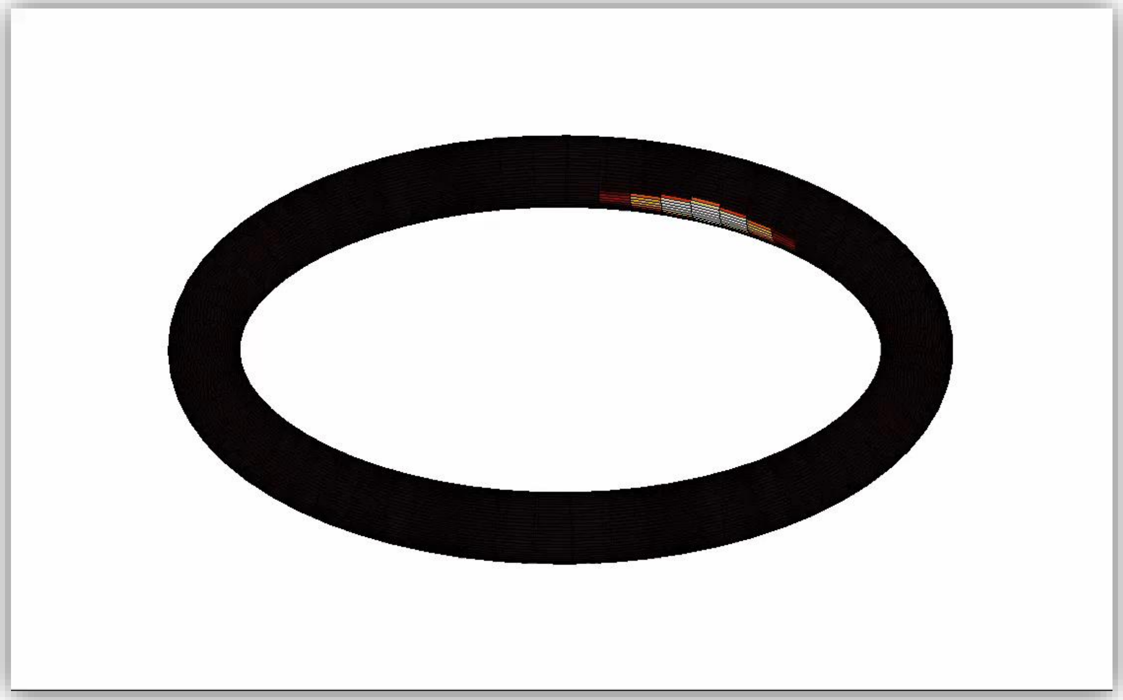
eigenvalues: -2.2749 $\quad -1 \quad$ 5.2749

eigenvectors: $\begin{pmatrix} 1 \\ 7.2749 \\ 1 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -0.2749 \\ 1 \end{pmatrix}$

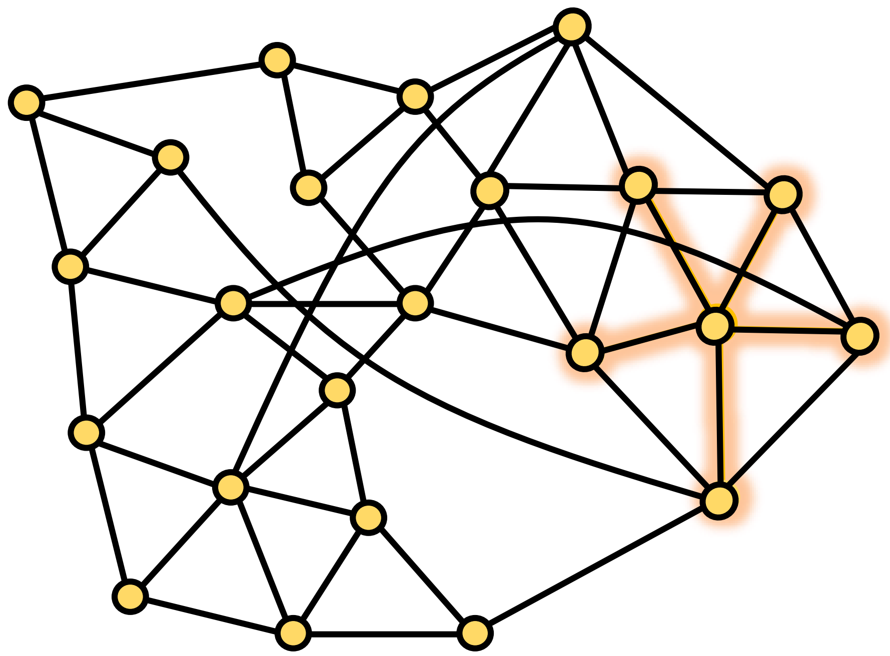
heat flow

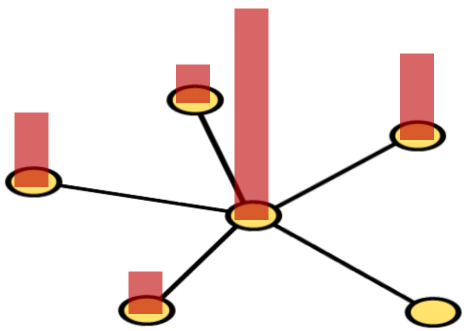
heat flow





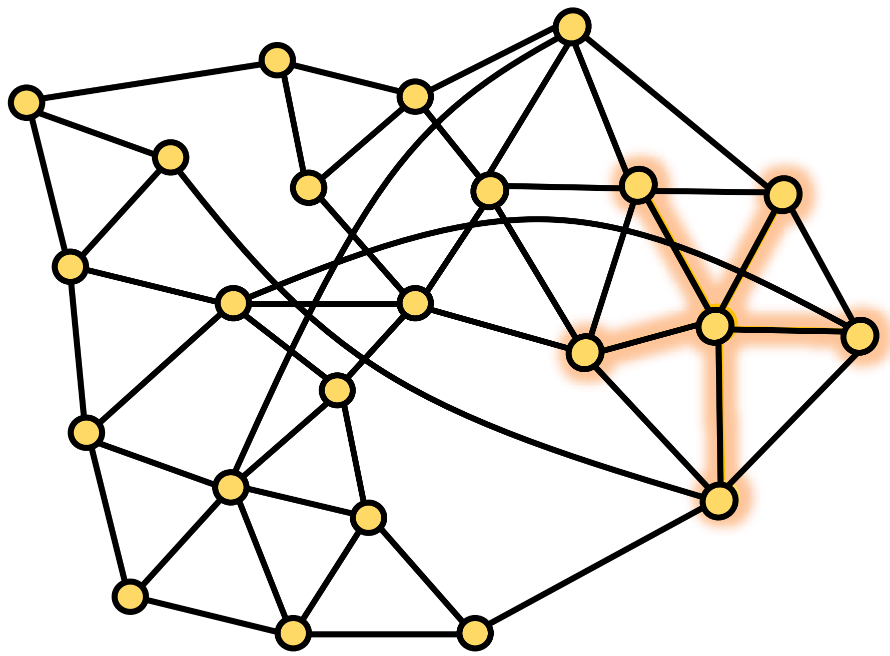
An undirected graph $G = (V, E)$





For now, assume that G is $\boldsymbol{d}$**-regular** for some number d.

An undirected graph $G = (V, E)$



$V = \{1, 2, \dots, n\}$

$u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$

Random walk matrix:

W is an $n \times n$ real symmetric matrix.

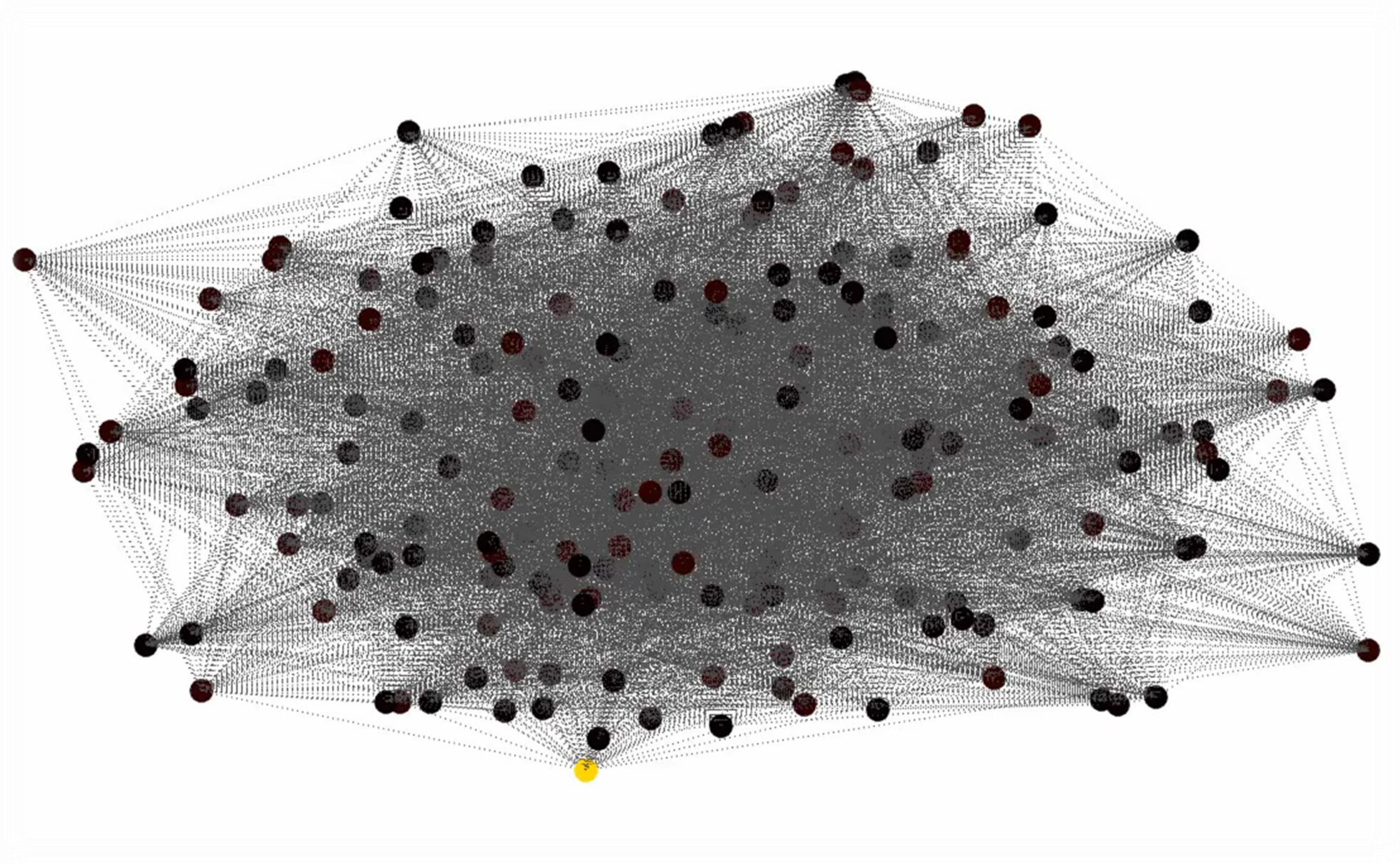
$$W_{ii} = \frac{1}{2}$$

$$W_{ij} = \frac{1}{2d} \qquad \{i, j\} \text{ an edge}$$

$$W_{ij} = 0 \qquad \{i, j\} \text{ not an edge}$$

$$(Wu)_i = \frac{1}{2} u_i + \frac{1}{2} \frac{1}{d} \sum_{j : \{i,j\} \in E} u_j$$

heat dispersion on a graph



evolution of the random walk / heat flow

$$u = (u_1, u_2, \dots, u_n)$$

$$Wu = \left(\sum_{i=1}^{n} W_{1,i} u_i \,, \sum_{i=1}^{n} W_{2,i}\, u_i, \dots, \sum_{i=1}^{n} W_{n,i}\, u_i \right)$$

$$W^2 u = \left(\sum_{i,j=1}^{n} W_{1,j} W_{j,i} u_i \,, \dots, \sum_{i,j=1}^{n} W_{n,j}\, W_{j,i}\, u_i \right)$$

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

$$Wu = \mu_1 \alpha_1 v_1 + \mu_2 \alpha_2 v_2 + \cdots + \mu_n \alpha_n v_n$$

$$W^2 u = \mu_1^2 \alpha_1 v_1 + \mu_2^2 \alpha_2 v_2 + \cdots + \mu_n^2 \alpha_n v_n$$

$$W^k u = \mu_1^k \alpha_1 v_1 + \mu_2^k \alpha_2 v_2 + \cdots + \mu_n^k \alpha_n v_n$$

eigenvalues/
eigenvectors of W

μ_1 v_1

μ_2 v_2

μ_n v_n

$$\mu_1 = 1$$

$$v_1 = \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}} \right)$$

evolution of the random walk / heat flow

$$u = (u_1, u_2, \dots, u_n)$$

$$Wu = \left(\sum_{i=1}^{n} W_{1,i} u_i \,, \sum_{i=1}^{n} W_{2,i}\, u_i, \dots, \sum_{i=1}^{n} W_{n,i}\, u_i \right)$$

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$$\mu_1 = 1$$

$$v_1 = \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}} \right)$$

$$u = (u_1, u_2, \ldots, u_n)$$

$$Wu = \left(\sum_{i=1}^{n} W_{1,i} u_i \, , \sum_{i=1}^{n} W_{2,i} \, u_i, \ldots, \sum_{i=1}^{n} W_{n,i} \, u_i \right)$$

$$W^2 u = \left(\sum_{i,j=1}^{n} W_{1,j} W_{j,i} u_i \, , \ldots, \sum_{i,j=1}^{n} W_{n,j} \, W_{j,i} \, u_i \right)$$

eigenvalues/
eigenvectors of W

μ_1	v_1
μ_2	v_2
μ_n	v_n

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

$$Wu = \mu_1 \alpha_1 v_1 + \mu_2 \alpha_2 v_2 + \cdots + \mu_n \alpha_n v_n$$

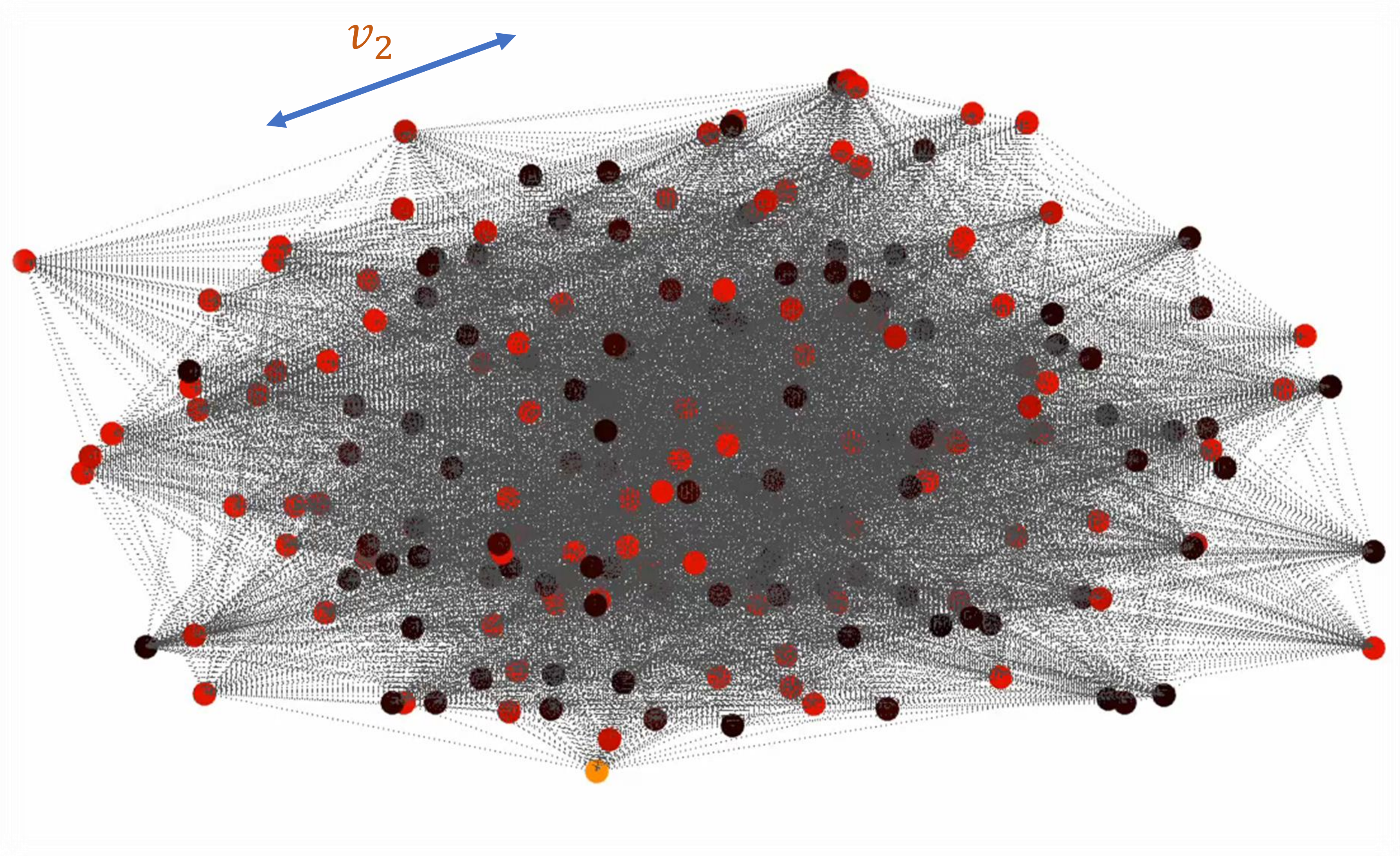
$$W^2 u = \mu_1^2 \alpha_1 v_1 + \mu_2^2 \alpha_2 v_2 + \cdots + \mu_n^2 \alpha_n v_n$$

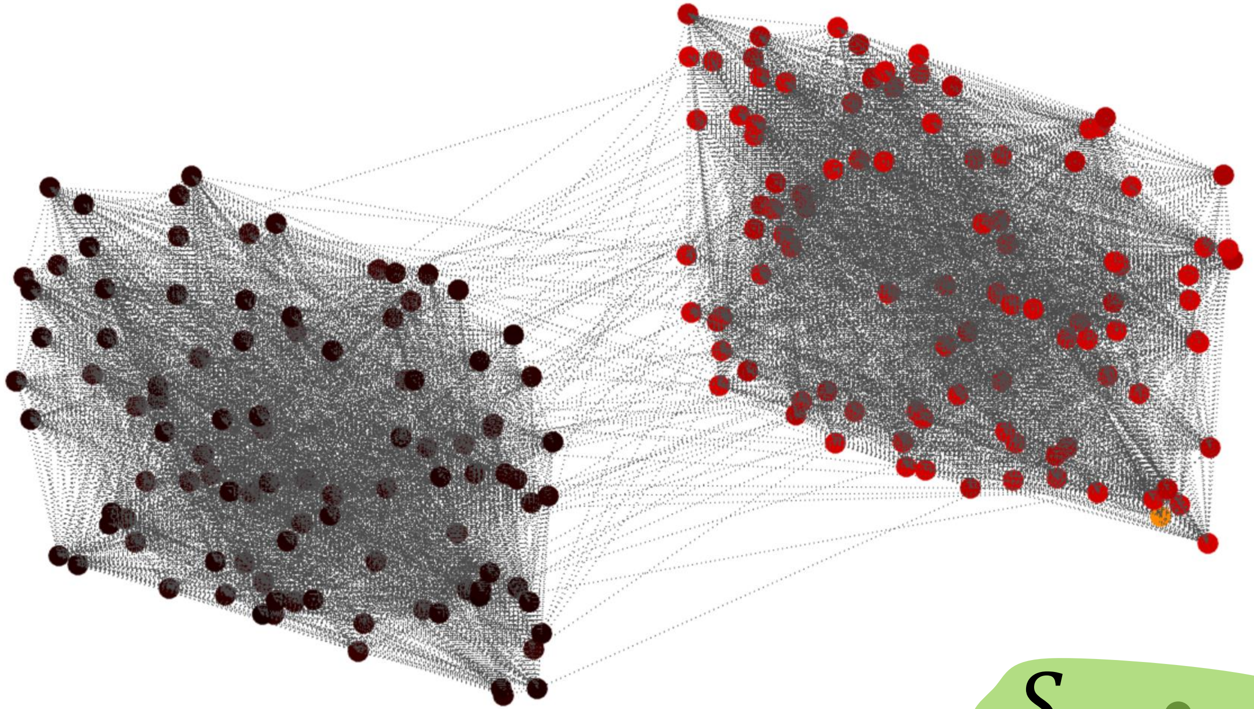
$$W^k u = \quad \alpha_1 v_1 + \mu_2^k \alpha_2 v_2 + \cdots + \mu_n^k \alpha_n v_n$$

$$\mu_1 = 1$$

$$v_1 = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right)$$

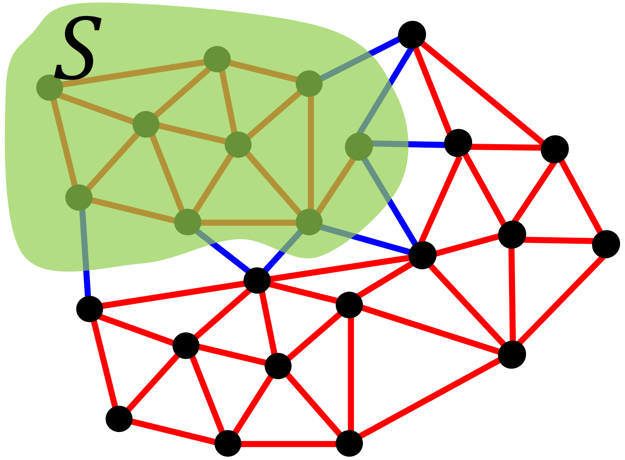
spectral embedding





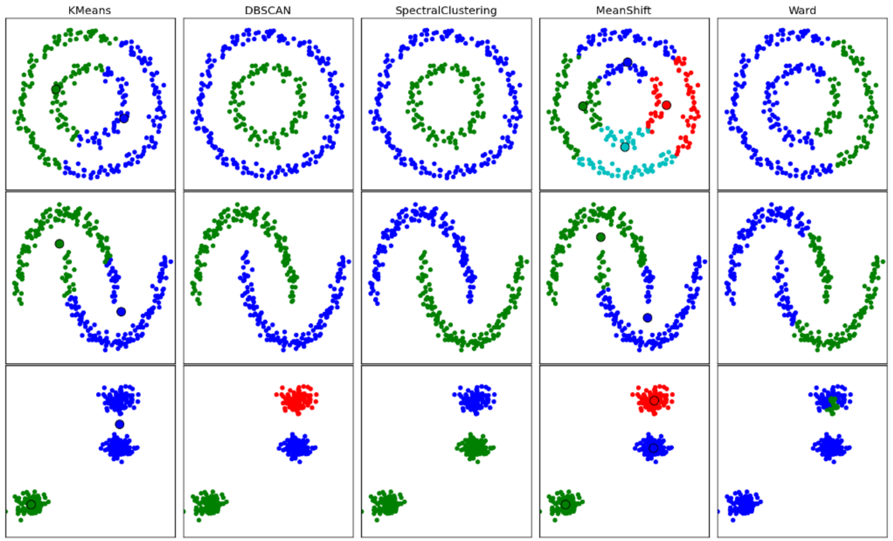
$$G = (V, E)$$

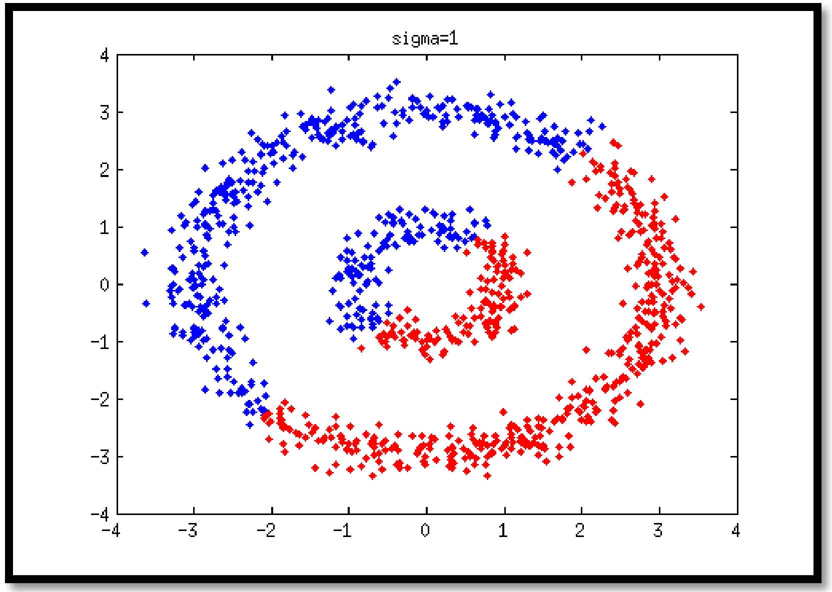
$$\Phi^*(G) = \min_{|S| \leq \frac{n}{2}} \Phi(S)$$

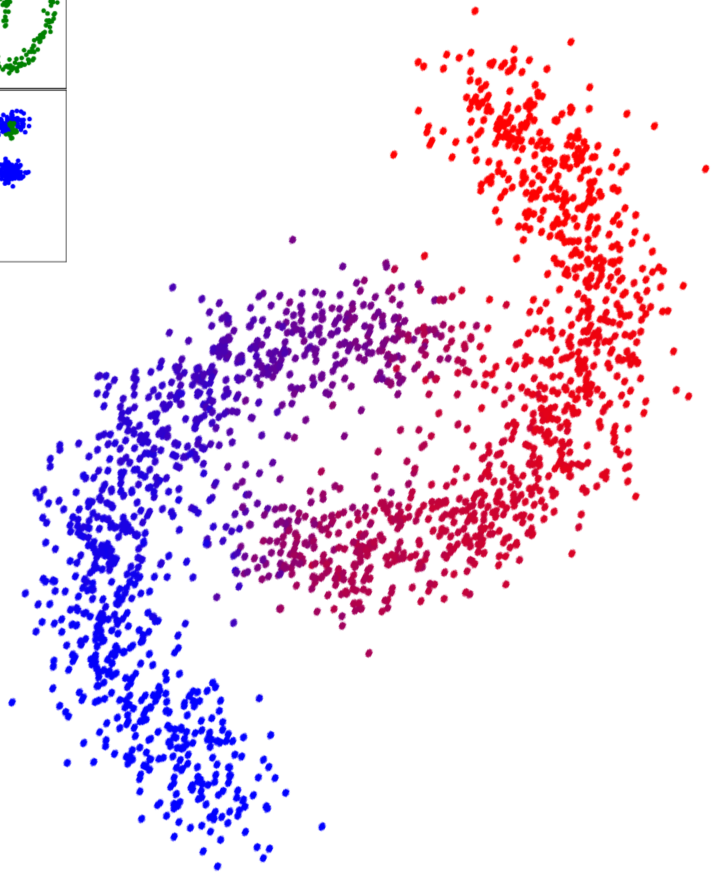


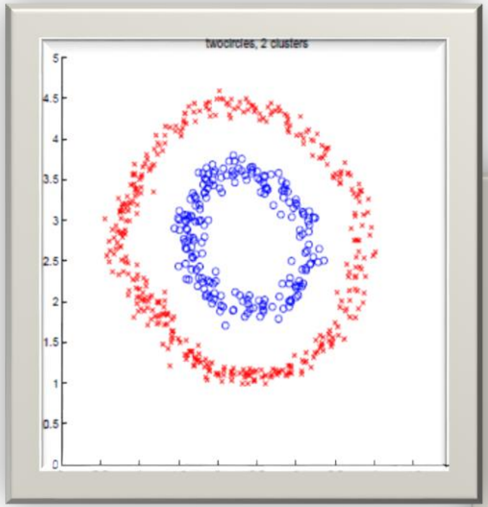
$$\Phi(S) = \frac{|E(S)|}{|S|}$$

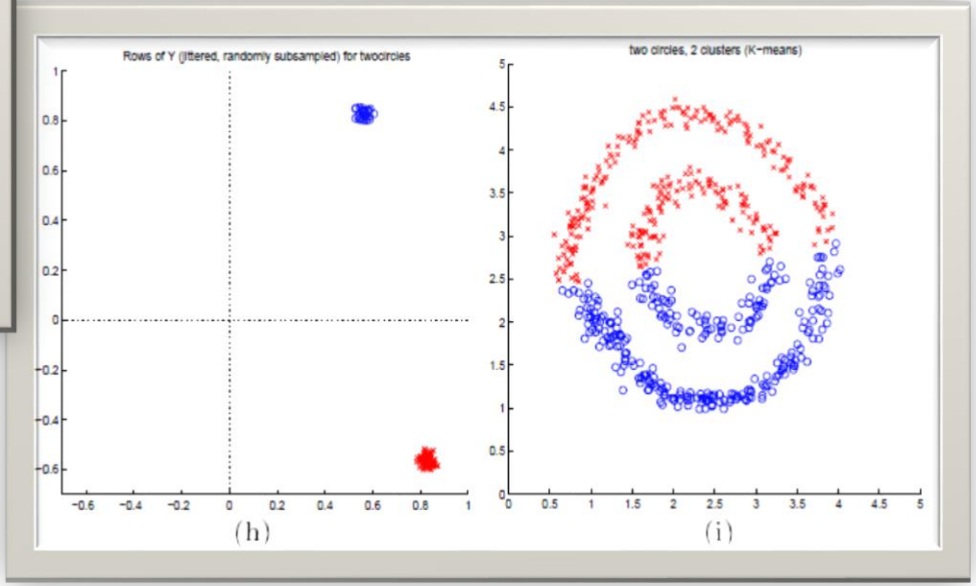
PCA cannot find non-linear structure

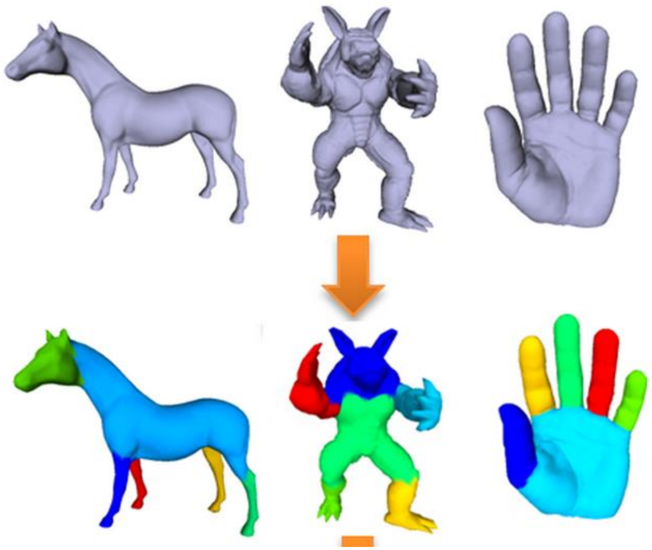






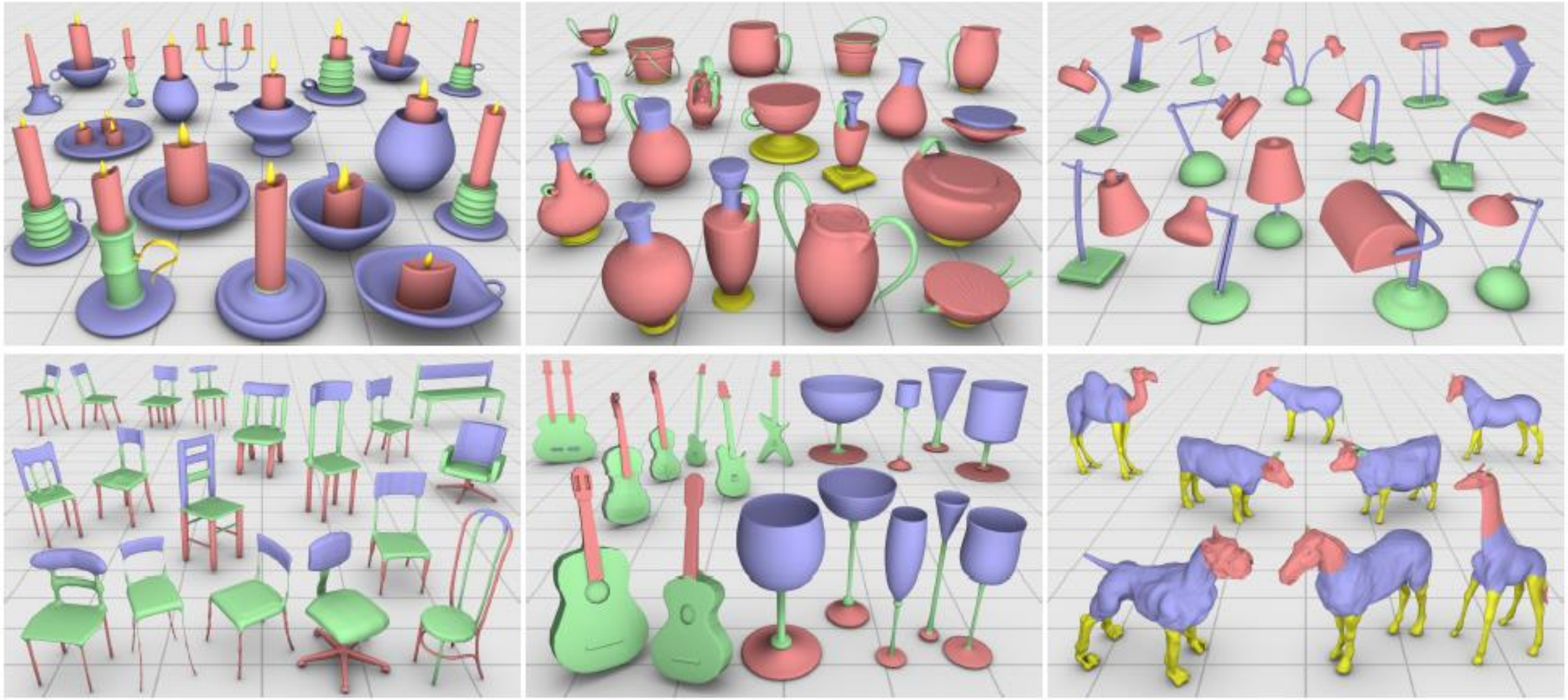






[photo credit: Ma-Wu-Luo-Feng 2011]

spectral partitioning can...



[photo credit: Sidi, et. al. 2011]