CSE 599I: Geometric embeddings (Winter 2007) **Instructor: James R. Lee**
Homework #2 **Due: Thursday, Feb. 15**

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1. **Ruminations on dimension.**

   (a) [5 points] Recall that if $G = (V, E)$ is an n-vertex graph, then its *vertex expansion* is defined by

   $$h(G) = \min_{S \subseteq V : |S| \leq \frac{n}{2}} \frac{|N(S)|}{|S|}$$

   where $N(S) = \{v \in V : d_G(v, S) \leq 1\}$. In lecture, we proved the following concentration bound for Lipschitz functions on G in terms of $h(G)$. For any map $f : G \to \mathbb{R}$, we have

   $$|\{x : |f(x) - \mathsf{med}(f)| \geq t\}| \leq n \cdot h(G)^{-\frac{t}{\|f\|_{\mathrm{Lip}}}}.$$

   The following is well-known: There exists a family of 3-regular graphs $G_n = (V_n, E_n)$ such that $|V_n| = n$ and $h(G_n) \geq 1.1$ for every $n \in \mathbb{N}$. Use this to prove that there exist n-point subsets $X_n \subseteq \ell_\infty^n$ such that any D-embedding into ℓ_∞^d requires $d = n^{\Omega(\frac{1}{D})}$. (A D-embedding is a map with distortion D.)

   (b) [5 points] Let $Q_k = \{0,1\}^k$ be equipped with the ℓ_1 (i.e. Hamming) metric. It is known that for $p \geq 2$, one has $c_p(Q_k) \geq \Omega\left(\sqrt{\frac{k}{p}}\right)$. Use this to prove that if there is a D-embedding of Q_k into ℓ_∞^d, then $d \geq 2^{\Omega(\frac{k}{D^2})}$. (Hint: Recall the lecture in which we proved a lower bound for dimension reduction in ℓ_1.)

   (c) [5 points] Let $w_1, w_2, \ldots, w_{2^k} \in \{-1,1\}^{2^k}$ be the rows of the $2^k \times 2^k$ Hadamard matrix. You will need the following fact: $\langle w_i, w_j \rangle = 0$ for $i \neq j$. Let $e_1, e_2, \ldots, e_{2^k}$ be the standard unit vectors in $\mathbb{R}^{2^k}$, and consider the set $A = \{0\} \cup \{w_1, \ldots, w_{2^k}\} \cup \{e_1, \ldots, e_{2^k}\}$. Show that when A is equipped with the ℓ_p metric for $1 \leq p \leq \infty$, then any linear operator $T : \ell_p^{2^k} \to L_2$ has $\mathrm{dist}(T|_A) \geq \left(\frac{|A|-1}{2}\right)^{|\frac{1}{p} - \frac{1}{2}|}$ where $T|_A$ is the map T restricted to A. (Hint: Find a way to compare $\sum_{i=1}^{2^k} \|w_i\|_p^2$ with $\sum_{i=1}^{2^k} \|T(e_i)\|_2^2$ using the distortion condition.)

   Use this result to show that any *linear* D-embedding of $(A, \|\cdot\|_p)$ into ℓ_p^d must have $D \geq \left(\frac{|A|-1}{2d}\right)^{|\frac{1}{p} - \frac{1}{2}|}$, i.e. there is no good linear dimension reduction in ℓ_p for $p \neq 2$.

PROBLEM 2 on REVERSE SIDE.

2. **Low-dimensional embeddings of trees into ℓ_1.**

In the first homework, we saw that every tree $T = (V, E)$ embeds isometrically into $\ell_1^{|E|}$. In this problem, we will show that with small distortion we can do much better.

(a) [10 points] Show that at least n dimensions are required to embed the $(2n + 1)$-point star metric (one center node and $2n$ leaves) isometrically into ℓ_1, and this is tight.

Let $T = (V, E)$ be an unweighted n-vertex tree with root $r \in V$. An *edge coloring of T* is a map $\chi : E \to \mathbb{N}$. A *monotone path in T* is a contiguous subset of some root-leaf path. An edge coloring $\chi : E \to \mathbb{N}$ is called *monotone* if every color class $\chi^{-1}(c)$ is a monotone path in T. (Note that $\chi^{-1}(c)$ will be an empty path for all but finitely many colors $c \in \mathbb{N}$. Empty paths are still monotone.)

(b) [5 points] Prove that any tree T has a monotone edge coloring so that every root-leaf path in T contains at most $O(\log n)$ distinct colors.

(c) [10 points] You will now prove that for any tree T, we have $c_1^{\mathrm{dom}(s)}(T) = O\left(\sqrt{\frac{\log n}{s}}\right)$.

Let $\chi : E \to \mathbb{N}$ be the coloring guaranteed by part (b), and let $s \geq 1$ be given. Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a set of i.i.d. random variables satisfying $\Pr(\varepsilon_i = 0) = 1 - \frac{1}{s}$, $\Pr(\varepsilon_i = 1) = \frac{1}{2s}$, and $\Pr(\varepsilon_i = -1) = \frac{1}{2s}$. We define a random embedding $F : X \to \mathbb{R}$ as follows. For a point $v \in V$, let $s_1, s_2, \ldots, s_k$ be the set of maximal χ-monochromatic segments on the path from the root to x. Then we set

$$F(x) = s \cdot \sum_{i=1}^{k} \mathsf{len}(s_i) \cdot \varepsilon_{\chi(s_i)},$$

where we have extended χ to the monochromatic segments $\{s_i\}_{i=1}^{k}$ in the natural way.

Analyze the embedding $F : V \to L_1$ to give the required bound on $c_1^{\mathrm{dom}(s)}(T)$. You may need to use Cauchy-Schwartz and Khintchine's inequality.

Conclude that T admits an $O(1)$-distortion embedding into $\ell_1^{O(\log n)^2}$. (Note: It is still an open problem whether the dimension can be improved to $O(\log n)$ even when T is the complete binary tree.)

(d) [Extra credit] Can you show that $c_1^{\mathrm{dom}(s)}(T_k) = O\left(\sqrt{\frac{\log k}{\log s}}\right)$ where T_k is the complete binary tree of height k? (I don't know if this is possible, but I also don't know anything that rules it out.)