

# 1 CSE 599I: Homework #2 solutions

1. (a) First, by Fréchet's embedding, we can choose  $X_n \subseteq \ell_\infty^n$  to be isometric to  $(G_n, d_{G_n})$ . Suppose that  $f : G_n \rightarrow \ell_\infty^d$  is a  $D$ -embedding. By scaling, we may assume that  $\|f\|_{\text{Lip}} \leq 1$  and  $\|f^{-1}\|_{\text{Lip}} \leq D$ .

Now, since  $G_n$  is 3-regular, there exist a constant  $\delta > 0$  such that for at least  $n^2/10$  pairs of nodes  $x, y \in G_n$ , we have  $d_{G_n}(x, y) \geq \delta \log n$ , hence letting

$$S = \left\{ \{x, y\} : x, y \in G_n \text{ and } \|f(x) - f(y)\|_\infty \geq \frac{\delta \log n}{D} \right\},$$

we have  $|S| \geq n^2/10$ .

Let  $\{f_i : G_n \rightarrow \mathbb{R}\}_{i=1}^d$  be the coordinates of the mapping  $f$ . Since  $\|f\|_{\text{Lip}} \leq 1$ , we have  $\|f_i\|_{\text{Lip}} \leq 1$  as well. On the one hand, for every  $\{x, y\} \in S$ , there must exist some  $i \in [d]$  for which  $|f_i(x) - f_i(y)| \geq \frac{\delta \log n}{D}$ . On the other hand, the number of pairs mapped  $\frac{\delta \log n}{D}$  apart by  $f_i$  is at most

$$n \cdot \left| \left\{ x : |f_i(x) - \text{med}(f_i)| \geq \frac{\delta \log n}{2D} \right\} \right| \leq n^2 h(G) e^{-\frac{\delta \log n}{2D}} \leq n^2 \cdot n^{-\Omega(1/D)}.$$

Thus we need at least  $d \geq n^{\Omega(1/D)}$  coordinates to handle all the pairs in  $S$ .

- (b) Use the fact that, by Hölder's inequality,  $\ell_\infty^d$  is  $O(1)$ -equivalent to  $\ell_{1+\log d}^d$  for all  $d \geq 1$ .
- (c) Write  $w_i = \sum_{j=1}^{2^k} w_{ij} e_j$ . Let  $T : \ell_p^{2^k} \rightarrow L_2$  be any linear operator which has  $\|T\|_{\text{Lip}} \leq L$  and  $\|T^{-1}\|_{\text{Lip}} \leq 1$ . Assume first of all that  $1 \leq p < 2$ . Then,

$$\begin{aligned} 2^{k(1+2/p)} &= \sum_{i=1}^{2^k} \|w_i\|_p^2 \leq \sum_{i=1}^{2^k} \|T w_i\|_2^2 = \sum_{i=1}^{2^k} \left\| \sum_{j=1}^{2^k} w_{ij} T(e_j) \right\|_2^2 \\ &= \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} \langle w_i, w_j \rangle \langle T(e_i), T(e_j) \rangle = 2^k \sum_{j=1}^{2^k} \|T(e_j)\|_2^2 \leq 4^k \cdot L^2, \end{aligned}$$

which implies that  $L \geq 2^{k(1/p-1/2)} = \left(\frac{|A|-1}{2}\right)^{1/p-1/2}$ . When  $p > 2$  apply the same reasoning, with the inequalities reversed.

The final result follows by noting that the norms on  $\ell_p^d$  and  $\ell_2^d$  are equivalent up to  $d^{|\frac{1}{p}-\frac{1}{2}|}$ .

2. (a) The upper bound here is obvious. For the lower bound, let  $f : S_{2n+1} \rightarrow \ell_1^d$  be an isometric embedding of the  $(2n+1)$ -star. Without loss, the center  $c$  is mapped to the origin. Consider the embedding of two leaves  $x$  and  $y$ :  $f(x) = (x_1, \dots, x_d)$  and  $f(y) = (y_1, \dots, y_d)$ . Then,

$$2 = \|f(x) - f(y)\|_1 = \sum_{i=1}^d |x_i - y_i| \leq \sum_{i=1}^d |x_i| + |y_i| = \|f(x) - f(c)\|_1 + \|f(y) - f(c)\|_1 = 2$$

We conclude that the middle inequality is tight, hence for every two leaves  $x, y$  and  $i \in [d]$ , we have  $x_i y_i \leq 0$ . This implies that for every coordinate  $i \in [d]$ , there can be at most two leaves whose  $i$ th coordinates are non-zero. On the other hand, obviously every leaf requires at least one non-zero coordinate. We conclude that  $d \geq n$ .

- (b) In fact, we can get a monotone coloring with at most  $1 + \log_2 \ell$  distinct colors on every root-leaf path, where  $\ell$  is the number of leaves in  $T$ . Simply choose a root-leaf path  $P$  such that upon the removal of  $P$  from  $T$ , each remaining connected component contains at most  $\ell/2$  of the original leaves. (This is easy to do—e.g. choose the  $\lfloor \ell/2 \rfloor$ th leaf in an in-order traversal of the tree.) Now color the remaining subtrees recursively.
- (c) It is clear that  $\Pr(\|F\|_{\text{Lip}} \leq s) = 1$  since it is easily checked that  $F$  never expands any edge by more than  $s$ . Now fix  $x, y \in T$  with least common ancestor  $u \in T$ , and let  $s_1, \dots, s_k$  and  $t_1, \dots, t_h$  be the set of maximal  $\chi$ -monochromatic segments on the path from  $u$  to  $x$  and  $y$ , respectively. Note that since  $\chi$  is monotone, the color classes of these segments are all pairwise disjoint. First, we have

$$\begin{aligned} \mathbb{E} [|F(x) - F(y)|] &\leq s \cdot \left( \sum_{i=1}^k \text{len}(s_i) \mathbb{E} |\varepsilon_{\chi(s_i)}| + \sum_{i=1}^h \text{len}(t_i) \mathbb{E} |\varepsilon_{\chi(t_i)}| \right) \\ &= \sum_{i=1}^k \text{len}(s_i) + \sum_{i=1}^h \text{len}(t_i) = d(x, y). \end{aligned}$$

On the other hand, observe that  $\varepsilon_i$  has the same distribution as  $|\varepsilon_i| \cdot \sigma_i$  where  $\{\sigma_i\}_{i=1}^\infty$  is a family of i.i.d.  $\pm 1$  Bernoulli random variables independent from the family  $\{\varepsilon_i\}_{i=1}^\infty$ . We use  $\mathbb{E}_\sigma$  and  $\mathbb{E}_\varepsilon$ , respectively, to denote expectations over these random variables. Using Khintchine's inequality, we have

$$\begin{aligned} \mathbb{E} [|F(x) - F(y)|] &= s \cdot \mathbb{E} \left| \sum_{i=1}^k \text{len}(s_i) \varepsilon_{\chi(s_i)} + \sum_{i=1}^h \text{len}(t_i) \varepsilon_{\chi(t_i)} \right| \\ &= s \cdot \mathbb{E}_\varepsilon \mathbb{E}_\sigma \left| \sum_{i=1}^k \text{len}(s_i) |\varepsilon_{\chi(s_i)}| \sigma_{\chi(s_i)} + \sum_{i=1}^h \text{len}(t_i) |\varepsilon_{\chi(t_i)}| \sigma_{\chi(t_i)} \right| \\ &\approx s \cdot \mathbb{E}_\varepsilon \sqrt{\sum_{i=1}^k \text{len}(s_i)^2 \varepsilon_{\chi(s_i)}^2 + \sum_{i=1}^h \text{len}(t_i)^2 \varepsilon_{\chi(t_i)}^2} \end{aligned}$$

Now, we let  $M = |\{i \in [k] : \varepsilon_{\chi(s_i)} \neq 0\}| + |\{i \in [h] : \varepsilon_{\chi(t_i)} \neq 0\}|$ . Observe that  $\mathbb{E} M = (k+h)/s$ . Let  $\mathcal{E}$  be the event that  $M \leq \max\left\{1, \frac{2(k+h)}{s}\right\}$ , and note that  $\Pr(\mathcal{E}) \geq \frac{1}{2}$ . Using Cauchy-Schwartz and  $k, h \leq O(\log n)$ , we have

$$\begin{aligned} \mathbb{E} [|F(x) - F(y)|] &\geq \Omega(1) \cdot s \cdot \Pr[\mathcal{E}] \cdot \mathbb{E} \left[ \sqrt{\sum_{i=1}^k \text{len}(s_i)^2 \varepsilon_{\chi(s_i)}^2 + \sum_{i=1}^h \text{len}(t_i)^2 \varepsilon_{\chi(t_i)}^2} \mid \mathcal{E} \right] \\ &\geq \frac{\Omega(1) \cdot s}{\max\{1, \sqrt{(\log n)/s}\}} \left( \sum_{i=1}^k \text{len}(s_i) \mathbb{E} \left[ \varepsilon_{\chi(s_i)}^2 \mid \mathcal{E} \right] + \sum_{i=1}^h \text{len}(t_i) \mathbb{E} \left[ \varepsilon_{\chi(t_i)}^2 \mid \mathcal{E} \right] \right) \\ &\geq \Omega(1) \cdot \min \left\{ 1, \sqrt{\frac{s}{\log n}} \right\} d(x, y), \end{aligned}$$

noting that  $\mathbb{E} [\varepsilon_{\chi(s_i)}^2 \mid \mathcal{E}] = \mathbb{E} [\varepsilon_{\chi(t_i)}^2 \mid \mathcal{E}] \approx 1/s$ .