

# Hardness of Learning Halfspaces with Noise

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## Abstract

*Learning an unknown halfspace (also called a perceptron) from labeled examples is one of the classic problems in machine learning. In the noise-free case, when a halfspace consistent with all the training examples exists, the problem can be solved in polynomial time using linear programming. However, under the promise that a halfspace consistent with a fraction  $(1 - \varepsilon)$  of the examples exists (for some small constant  $\varepsilon > 0$ ), it was not known how to efficiently find a halfspace that is correct on even 51% of the examples. Nor was a hardness result that ruled out getting agreement on more than 99.9% of the examples known.*

*In this work, we close this gap in our understanding, and prove that even a tiny amount of worst-case noise makes the problem of learning halfspaces intractable in a strong sense. Specifically, for arbitrary  $\varepsilon, \delta > 0$ , we prove that given a set of examples-label pairs from the hypercube a fraction  $(1 - \varepsilon)$  of which can be explained by a halfspace, it is NP-hard to find a halfspace that correctly labels a fraction  $(1/2 + \delta)$  of the examples.*

*The hardness result is tight since it is trivial to get agreement on  $1/2$  the examples. In learning theory parlance, we prove that weak proper agnostic learning of halfspaces is hard. This settles a question that was raised by Blum et al in their work on learning halfspaces in the presence of random classification noise [7], and in some more recent works as well. Along the way, we also obtain a strong hardness for another basic computational problem: solving a linear system over the rationals.*

## 1 Introduction

This work deals with the complexity of two fundamental optimization problems: solving a system of linear equations over the rationals, and learning a halfspace from labeled examples. Both these problems are “easy” when a perfect solution exists. If the linear system is satisfiable, then a satisfying assignment can be found in polynomial time by Gaussian Elimination. If a halfspace consistent with all the examples exists, then one can be found using linear program-

ming. A natural question that arises is thus the following: If no perfect solution exists, but say a solution satisfying 99% of the constraints exists, can we find a solution that is nearly as good (say, satisfies 90% of the constraints)?

This question has been considered for both these problems (and many others), but our focus here is the case when the instance is near-satisfiable (or only slightly noisy). That is, for arbitrarily small  $\varepsilon > 0$ , a solution satisfying at least a fraction  $(1 - \varepsilon)$  of the constraints is promised to exist, and our goal is to find an assignment satisfying as many constraints as possible. Sometimes, the problem is easier to solve on near-satisfiable instances — notable examples being the Max 2SAT and Max HornSAT problems. For both of these it is possible to find, in polynomial time, an assignment satisfying a fraction  $1 - f(\varepsilon)$  of the clauses  $f(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$  given a  $(1 - \varepsilon)$ -satisfiable instance [21]. Our results show that in the case of solving linear systems or learning halfspaces, we are not so lucky and finding any non-trivial assignment for  $(1 - \varepsilon)$ -satisfiable instances is NP-hard. We describe the context and related work as well as our results for the two problems in their respective subsections below.

Before doing that, we would like to stress that for problems admitting a polynomial time algorithm for satisfiability testing, hardness results of the kind we get, with gap at the right location (namely completeness  $1 - \varepsilon$  for any desired  $\varepsilon > 0$ ), tend to be hard to get. The most celebrated example in this vein is Håstad’s influential result [13] which shows that given a  $(1 - \varepsilon)$ -satisfiable instance of linear equations modulo a prime  $p$ , it is NP-hard to satisfy a fraction  $(\frac{1}{p} + \delta)$  fraction of them (note that one can satisfy a fraction  $\frac{1}{p}$  of the equations by simply picking a random assignment). Recently, Feldman [10] established a result in this vein in the domain of learning theory. He proved the following strong hardness result for weak learning monomials: given a set of example-label pairs a  $(1 - \varepsilon)$  fraction of which can be explained by a monomial, it is hard to find a monomial that correctly labels a fraction  $(1/2 + \delta)$  of the examples. Whether such a strong negative result holds for learning halfspaces also, or whether the problem admits a non-trivial weak learning algorithm is mentioned as a notable open question in [10], and this was also posed by Blum, Frieze, Kannan, and Vempala [7] almost 10 years

\*Research supported by NSF Career Award CCF-0343672, a Sloan Research Fellowship, and a Packard Foundation Fellowship.

†Research supported in part by NSF CCF-0343672,

ago. In this work, we establish a tight hardness result for this problem. We prove that given a set of example-label pairs a fraction  $(1 - \varepsilon)$  of which can be explained by a halfspace, finding a halfspace with agreement better than  $1/2$  is NP-hard. This result was also established independently (for real-valued examples) in [11].

### 1.1 Solving linear systems

We prove the following hardness result for solving noisy linear systems over rationals: For every  $\varepsilon, \delta > 0$ , given a system of linear equations over  $\mathbb{Q}$  which is  $(1 - \varepsilon)$ -satisfiable, it is NP-hard to find an assignment that satisfies more than a fraction  $\delta$  of the equations. As mentioned above, a result similar to this was shown by Håstad [13] for equations over a large finite field. But this does not seem to directly imply any result over rationals. Our proof is based on a direct reduction from the Label Cover problem. While by itself quite straightforward, this reduction is a stepping stone to our more complicated reduction for the problem of learning halfspaces.

The problem of approximating the number of satisfied equations in an unsatisfiable system of linear equations over  $\mathbb{Q}$  has been studied in the literature under the label MAX-SATISFY and strong hardness of approximation results have been shown in [4, 9]. In [9], it is shown that unless  $\text{NP} \subset \text{BPP}$ , for every  $\varepsilon > 0$ , MAX-SATISFY cannot be approximated within a ratio of  $n^{1-\varepsilon}$  where  $n$  is the number of equations in the system. (On the algorithmic side, the best approximation algorithm for the problem, due to Halldorsson [12], achieves ratio  $O(n/\log n)$ .) The starting point of the reductions in these hardness results is a system that is  $\rho$ -satisfiable for some  $\rho$  bounded away from 1 (in the completeness case), and this only worsens when the gap is amplified.

For the complementary objective of minimizing the number of unsatisfied equations, a problem called MIN-UNSATISFY, hardness of approximation within ratio  $2^{\log^{0.99} n}$  is shown in [4] (see also [3]). In particular, for arbitrarily large constants  $c$ , the reduction of Arora *et al* [4] shows NP-hardness of distinguishing between  $(1 - \gamma)$ -satisfiable instances and instances that are at most  $(1 - c\gamma)$ -satisfiable, for some  $\gamma$ . One can get a hardness result for MAX-SATISFY like ours by applying a standard gap amplification method to such a result (using a  $O(1/\gamma)$ -fold product construction), provided  $\gamma = \Omega(1)$ . As presented in [4], however, their reduction works with  $\gamma = o(1)$ . It is not difficult to modify their reduction to have  $\gamma = \Omega(1)$ . Our reduction is somewhat different, and serves as a warm-up for the reduction for learning halfspaces, which we believe puts together an interesting combination of techniques.

### 1.2 Halfspace learning

Learning halfspaces (also called *Perceptrons* or *linear threshold functions*) is one of the oldest problems

in machine learning. Formally, a halfspace on variables  $x_1, \dots, x_n$  is a Boolean function  $I[w_1x_1 + w_2x_2 + \dots + w_nx_n \geq \theta]$  for reals  $w_1, \dots, w_n, \theta$  (here  $I[E]$  is the indicator function for an event  $E$ ). For definiteness, let us assume that variables  $x_i$  are Boolean, that is, we are learning functions over the hypercube  $\{0, 1\}^n$ . In the absence of noise, one can formulate the problem of learning a halfspace as a linear program and thus solve it in polynomial time. In practice, simple incremental algorithms such as the famous Perceptron Algorithm [1, 18] or the Winnow algorithm [17] are often used.

Halfspace-based learning algorithms are popular in theory and practice, and are often applied to labeled examples sets which are not separable by a halfspace. Therefore, an important question that arises and has been studied in several previous works is the following: what can one say about the problem of learning halfspaces in the presence of noisy data that does not obey constraints induced by an unknown halfspace?

In an important work on this subject, Blum, Frieze, Kannan, and Vempala [7] gave a PAC learning algorithm for halfspaces in the presence of *random classification noise*. Here the assumption is that the examples are generated according to a halfspace, except with a certain probability  $\eta < 1/2$ , the label of each example is independently flipped. The learning algorithm in [7] outputs as hypothesis a decision list of halfspaces. Later, Cohen [8] gave a different algorithm for random classification noise where the output hypothesis is also a halfspace. (Such a learning algorithm whose output hypothesis belongs to the concept class being learned is called a *proper learner*.) These results applied to PAC learning with respect to arbitrary distributions, but assume a rather “benign” noise model that can be modeled probabilistically.

For learning in more general noise models, an elegant framework called *agnostic learning* was introduced by Kearns *et al* [16]. Under agnostic learning, the learner is given access to labeled examples  $(x, y)$  from a fixed distribution  $\mathcal{D}$  over example-label pairs  $X \times Y$ . However, there is no assumption that the labels are generated according to a function from specific concept class, namely halfspaces in our case. The goal of the learner is to output a hypothesis  $h$  whose accuracy with respect to the distribution is close to that of the best halfspace — in other words the hypothesis does nearly as well in labeling the examples as the best halfspace would.

In a recent paper [14], Kalai, Klivans, Mansour and Servedio gave an efficient agnostic learning algorithm for halfspaces when the marginal  $\mathcal{D}_X$  on the examples is the uniform distribution on the hypercube, or any log-concave distribution on the sphere  $S^{n-1}$ . For any desired  $\varepsilon > 0$ , their algorithm produces a hypothesis  $h$  with error rate  $\Pr_{(x,y) \in \mathcal{D}}[h(x) \neq y]$  at most  $\text{opt} + \varepsilon$  if the best halfspace

has error rate opt. Their output hypothesis itself is not a halfspace but rather a higher degree threshold function.

When the accuracy of the output hypothesis is measured by the fraction of agreements (instead of disagreements or mistakes), the problem is called *co-agnostic learning*. The combinatorial core of co-agnostic learning is the *Maximum Agreement* problem: Given a collection of example-label pairs, find the hypothesis from the concept class (a halfspace in our case) that correctly labels the maximum number of pairs. Indeed, it is well-known that an efficient  $\alpha$ -approximation algorithm to this problem exists iff there is an efficient co-agnostic proper PAC-learning algorithm that produces a halfspace that has agreement within a factor  $\alpha$  of the best halfspace.

The Maximum Agreement for Halfspaces problem, denoted HS-MA, was shown to be NP-hard to approximate within some constant factor for the  $\{0, 1, -1\}$  domain in [3, 6] (the factor was  $261/262 + \varepsilon$  in [3] and  $415/418 + \varepsilon$  in [6]). The best known hardness result prior to work was due to Bshouty and Burroughs, who showed an inapproximability factor of  $84/85 + \varepsilon$ , and their result applied also for the  $\{0, 1\}$  domain. For instances where a halfspace consistent with  $(1 - \varepsilon)$  of the examples exists (the setting we are interested in), an inapproximability result for HS-MA was *not* known for any fixed factor  $\alpha < 1$ . For the complementary objective of minimizing disagreements, hardness of approximating within a ratio  $2^{O(\log^{1-\varepsilon} n)}$  is known [4, 3]. The problem of whether an  $\alpha$ -approximation algorithm exists for HS-MA for some  $\alpha > 1/2$ , i.e., whether a weak proper agnostic learning algorithm for halfspaces exists, remained open. This question was also highlighted in Feldman's recent work [10] which proved that weak agnostic learning of monomials was hard.

In this paper, we prove that no  $(1/2 + \delta)$ -approximation algorithm exists for HS-MA for any  $\delta > 0$  unless  $P = NP$ . Specifically, for every  $\varepsilon, \delta > 0$ , it is NP-hard to distinguish between instances of HS-MA where a halfspace agreeing on a  $(1 - \varepsilon)$  fraction of the example-label pairs exists and where no halfspace agrees on more than a  $(1/2 + \delta)$  fraction of the example-label pairs. Our hardness result holds for examples drawn from the hypercube. Our result indicates that for *proper* learning of halfspaces in the presence of even small amounts of noise, one needs to make assumptions about the nature of noise (such as random classification noise studied in [7]) or about the distribution of the example-label pairs (such as uniform marginal distribution on examples as in [14]).

A similar hardness result was proved independently by Feldman *et al* [11] for the case when the examples are drawn from  $\mathbb{R}^n$ . In contrast, our proof works when the data points are restricted to the hypercube  $\{0, 1\}^n$ , which is the natural setting for a Boolean function. Much of the complexity of our reduction stems from ensuring that the

examples belong to the hypercube.

## 2 Preliminaries

The first of the two problems studied in this paper is the following:

**Definition 2.1.** For constants  $c, s$ , satisfying  $0 \leq s \leq c \leq 1$ , LINEQ-MA( $c, s$ ) refers to the following Promise problem: Given a set of linear equations over variables  $X = \{x_1, \dots, x_n\}$ , with coefficients over  $Q$ , distinguish between the following two cases:

- There is an assignment of values to the variables  $X$ , that satisfies at least a fraction  $c$  of the equations.
- Every assignment satisfies less than a fraction  $s$  of the equations.

In the problem of learning a halfspace to represent a boolean function, the input consists of a set of positive and negative examples all from the boolean hypercube. These examples are embedded in the real  $n$ -dimensional space  $\mathbb{R}^n$ , by some natural embedding. The objective is to find a hyperplane in  $\mathbb{R}^n$  that separates, the positive and the negative examples.

**Definition 2.2.** Given two disjoint multisets of vectors  $S^+, S^- \subset \{-1, 1\}^n$ , a vector  $a \in \mathbb{R}^n$ , and a threshold  $\theta$ , the agreement of the halfspace  $a \cdot v \geq \theta$  with  $(S^+, S^-)$  is defined to be the quantity

$$|\{v | v \in S^+, a \cdot v \geq \theta\}| + |\{v | v \in S^-, a \cdot v < \theta\}|.$$

where the cardinalities are computed, by counting elements with repetition. In the HS-MA problem, the goal is to find  $a, \theta$  such that the halfspace  $a \cdot v \geq \theta$  maximizes this agreement.

The learning problem in the  $\{-1, 1\}^n$  embedding can be shown to be equivalent to the learning problem on most natural embeddings such as  $\{0, 1\}^n$ . Hence there is no loss of generality in assuming the embedding to be  $\{-1, 1\}^n$ . Further, our hardness result holds even if both the inequalities  $\{\geq, <\}$  are replaced by strict inequalities  $\{>, <\}$ .

To study the hardness of approximating HS-MA, we define the following promise problem:

**Definition 2.3.** For constants  $c, s$  satisfying  $0 \leq s \leq c \leq 1$ , define HS-MA( $c, s$ ) to be the following Promise problem: Given multisets of positive and negative examples  $S^+, S^- \subset \{-1, 1\}^n$  distinguish between the following two cases:

- There is a halfspace  $a \cdot v \geq \theta$  that has agreement at least  $c|S^+ \cup S^-|$  with  $(S^+, S^-)$ .
- Every halfspace has agreement less than  $s|S^+ \cup S^-|$  with  $(S^+, S^-)$ .

The hardness results in this paper are obtained by reductions from the Label Cover problem defined below.

**Definition 2.4.** An instance of LABELCOVER( $c, s$ ) represented as  $\Gamma = (U, V, E, \Sigma, \Pi)$ , consists of a bipartite graph over node sets  $U, V$  with the edges  $E$  between them, such that all nodes in  $U$  are of the same degree. Also part of the instance is a set of labels  $\Sigma$ , and a set of mappings  $\pi_e : \Sigma \rightarrow \Sigma$  for each edge  $e \in E$ . An assignment  $A$  of labels to vertices is said to satisfy an edge  $e = (u, v)$ , if  $\pi_e(A(u)) = A(v)$ . The problem is to distinguish between the following two cases:

- There exists an assignment  $A$  that satisfies at least a fraction  $c$  of the edge constraints  $\Pi$
- Every assignment satisfies less than a fraction  $s$  of the constraints in  $\Pi$ .

The reductions in this paper use the following inapproximability result for Label Cover.

**Theorem 2.5.** [20, 5] There exists an absolute constant  $\gamma > 0$  such that for all large enough  $R$ , the gap problem LABELCOVER( $1, \frac{1}{R^\gamma}$ ) is NP-hard, where  $R = |\Sigma|$  is the size of the alphabet.

Throughout this paper, we use the letter  $E$  to denote a linear equation/function, with coefficients  $\{0, 1, -1\}$ . For a linear function  $E$ , we use  $V(E)$  to denote the set of variables with non-zero coefficients in  $E$ . Further the evaluation  $E(A)$  for an assignment  $A$  of real values to the variables is the real value obtained on substituting the assignment in the equation  $E$ . Hence, an assignment  $A$  satisfies the equation  $E$  if  $E(A) = 0$ . For the purposes of the proof, we make the following definitions.

**Definition 2.6.** An equation tuple  $T$  consists of a set of linear equations  $E_1, \dots, E_k$  and a linear function  $E$  called the scaling factor.

**Definition 2.7.** A tuple  $T = (\{E_1, E_2, \dots, E_k\}, E)$  is said to be disjoint if the sets of variables  $V(E_i)$ ,  $1 \leq i \leq k$ , and  $V(E)$  are all pairwise disjoint. An equation tuple is said to be of constant arity, if the arity of each of its equations and the scaling factor are bounded by a constant.

**Definition 2.8.** An assignment  $A$  is said to satisfy an equation tuple  $T = (\{E_1, \dots, E_k\}, E)$ , if the scaling factor is positive, i.e.,  $E(A) > 0$ , and for every  $i$ ,  $1 \leq i \leq k$ ,  $E_i(A) = 0$ . For  $\beta \geq 0$ , an assignment  $A$  is said to  $\beta$ -satisfy an equation tuple  $T$  if  $E(A) > 0$  and for each  $1 \leq i \leq k$ ,  $|E_i(A)| \leq \beta \cdot |E(A)|$ . Note that 0-satisfying an equation tuple is the same as satisfying it.

**Definition 2.9.** An assignment  $A$  is said to be  $C$ -close to  $\beta$ -satisfying an equation tuple  $T = (\{E_1, \dots, E_k\}, E)$ , if

$E(A) > 0$  and  $|E_i(A)| > \beta|E(A)|$  for at most  $C$  values of  $i$ ,  $1 \leq i \leq k$ . An assignment is said to be  $C$ -far from  $\beta$ -satisfying an equation tuple  $T$  if it is not  $C$ -close to  $\beta$ -satisfying  $T$ .

### 3 Overview of the Proof

Both the hardness results use a reduction from the Label Cover problem. The proof of hardness of HS-MA proceeds in three stages as described below.

In the first stage the label cover problem is reduced to a set of equation tuples  $\mathcal{T}$  using Verifier I such that for a NO instance of label cover, any assignment  $A$  can  $\beta$ -satisfy a very tiny fraction of tuples in  $\mathcal{T}$ . However the tuples  $T \in \mathcal{T}$  are not disjoint.

In the second stage, Verifier II takes as input the set  $\mathcal{T}$  and creates a set of equation tuples  $\mathcal{T}'$ . The tuples in  $\mathcal{T}'$  are disjoint, they are all over the same set of variables, and each variable appears in exactly one equation of every tuple. Further, in the soundness case, almost all tuples are at least  $C$ -far from being  $\varepsilon$ -satisfied. Verifier II thus plays two roles: (i) it makes the equations in each tuple have disjoint support, and (ii) in the soundness case, every assignment not just fails to  $\varepsilon$ -satisfy most of the tuples, but is in fact  $C$ -far from  $\varepsilon$ -satisfying most of the tuples. Both these facts are exploited by Verifier III in the third stage.

Verifier III distinguishes, by checking suitable inequalities, between the cases when an assignment  $A$  satisfies a tuple  $T$  and when it is  $C$ -far from  $\beta$ -satisfying  $T$ . The inequalities are based on a random linear combination of the equations of the tuple with  $\pm 1$  coefficients (the random choice is made from a small sample space of vectors with  $\pm 1$  coefficients). In the completeness analysis, if all equations are satisfied, i.e., evaluate to 0 on  $A$ , then any  $\pm 1$  combination also vanishes. In the soundness analysis, most tuples have at least  $C$  equations with non-trivial absolute value, and this implies that their linear combination is unlikely to be small (a careful choice of the sample space of linear combinations is crucial to conclude this).

Each of the inequalities checked by Verifier III has all the variables with coefficients  $\{-1, 1\}$ , and has a common variable (a threshold  $\theta$ ) on the right hand side. Hence the checks made by the combined verifier correspond naturally to training examples in the learning problem.

For the hardness of LINEQ-MA, the set of tuples  $\mathcal{T}$  output by Verifier I are rather easily converted in to a set of equations. This is achieved by creating several equations for each equation tuple  $T \in \mathcal{T}$ , such that a large fraction of these are satisfied if and only if  $T$  is satisfied.

### 4 Verifier I

Let  $(U, V, E, \Sigma, \Pi)$  be an instance of Label Cover with  $|\Sigma| = R$ . This verifier produces a set of equation tuples,

which are tested using Verifier II. The equation tuples have variables  $u_1, \dots, u_R$  for each vertex  $u \in U \cup V$ . The solution that we are targeting is an encoding of the assignment to the label cover instance. So if a vertex  $u$  is assigned the label  $i$  by an assignment  $A$ , then we want  $u_i = 1$  and  $u_j = 0$  for  $j \neq i, 1 \leq j \leq R$ . We construct an equation tuple for every  $t$ -tuple of variables corresponding to vertices in  $U$ , for a suitable parameter  $t$  that will be chosen shortly.

For each  $t$ -tuple  $X$  of variables corresponding to vertices in  $U$ , construct the equation tuple  $T$  as follows.

- $\mathcal{P}_1$ : For every pair of vertices  $u, v \in U \cup V$ , an equation

$$\sum_{i=1}^R u_i - \sum_{j=1}^R v_j = 0$$

- $\mathcal{P}_2$ : For each edge  $e = (u, v) \in E$  the label cover constraint for the edge

$$\sum_{j \in \pi_e^{-1}(i)} u_j - v_i = 0 \text{ for all } 1 \leq i \leq R$$

- $\mathcal{P}_3$ : For each variable  $v \in X, v = 0$
- The scaling factor is  $\mathcal{P}_4$ :  $\sum_{i=1}^R u_i$  for some fixed vertex  $u \in U \cup V$

Output the tuple  $T = (\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3, \mathcal{P}_4)$

**Theorem 4.1.** For every  $\delta_1, \varepsilon_1 > 0$  there exists a sufficiently large  $R = R(\varepsilon_1, \delta_1)$  such that if  $\Gamma = (U, V, E, \Sigma, \Pi)$  is an instance of label cover with  $|\Sigma| = R$  then with the choice of  $\beta' = \frac{1}{R^3}$  the following holds:

- If  $\Gamma$  is satisfiable, then there is an assignment  $A$  that satisfies at least  $1 - \varepsilon_1$  fraction of the output tuples.
- If no assignment to  $\Gamma$  satisfies a fraction  $\frac{1}{R^\gamma}$  of the edges, then every assignment  $A$   $\beta'$ -satisfies less than a fraction  $\delta_1$  of the output tuples.

*Proof.* Let us choose parameters  $c_0 = \ln(1/\delta_1)$  and  $t = 4c_0 R^{1-\gamma}$ , for a sufficiently large  $R$ . We present the completeness and soundness arguments in turn.

**Completeness:** Given an assignment  $A$  to the Label Cover instance, that satisfies all the edges, the corresponding integer solution satisfies :

- All equations in  $\mathcal{P}_1$  and  $\mathcal{P}_2$ .
- $(1 - \frac{1}{R})$  fraction of the equations in  $\mathcal{P}_3$  for each edge  $e$ .

Since  $t$  equations of the form  $\mathcal{P}_3$  are present in each tuple, the assignment  $A$  satisfies at least  $(1 - \frac{1}{R})^t > 1 - \varepsilon_1$  of the tuples for large enough  $R$ .

**Soundness:** Suppose there is an assignment  $A$  that  $\beta'$ -satisfies at least a fraction  $\delta_1$  of the tuples generated. Clearly  $A$  must  $\beta'$ -satisfy all the equations  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , since they are common to all the tuples. Further by definition of  $\beta'$ -satisfaction, the scaling factor  $\mathcal{P}_4(A) > 0$ . Normalize the assignment  $A$  such that the scaling factor  $\mathcal{P}_4$  is equal to 1. As all the equations in  $\mathcal{P}_1$  are  $\beta'$ -satisfied, we get

$$1 - \beta' \leq \sum_{i=1}^R v_i \leq 1 + \beta', \text{ for all } v \in U \cup V \quad (1)$$

Further, we claim that the assignment  $A$   $\beta'$ -satisfies at least a fraction  $(1 - \frac{c_0}{t})$  of the equations in  $\mathcal{P}_3$ . Otherwise, with  $t$  of these equations belonging to every tuple, less than  $(1 - \frac{c_0}{t})^t < \delta_1$  tuples will be  $\beta'$ -satisfied by  $A$ . Recall that all vertices in  $U$  have same degree. Hence by an averaging argument, for at least half the edges  $e = (u, v)$ , at least  $(1 - \frac{2c_0}{t})$  of the constraints  $u_i = 0$  are  $\beta'$ -satisfied. Let us call these edges *good*.

For every vertex  $w$ , define the set of labels  $\text{Pos}$  as follows,

$$\begin{aligned} \text{Pos}(u) &= \{i \in \Sigma \mid u_i \geq 8\beta'\} \text{ if } u \in U \\ \text{Pos}(v) &= \{j \in \Sigma \mid v_j \geq 8\beta'(R+1)\} \text{ if } v \in V \end{aligned}$$

The set  $\text{Pos}(w)$  is non-empty for each vertex  $w \in U \cup V$ , because otherwise  $\sum_{i=1}^R w_i < 8\beta'(R+1) \cdot R \leq 1 - \beta'$ , a contradiction to (1). Further if  $e = (u, v)$  is a *good* edge then for at least  $1 - \frac{2c_0}{t}$  of the labels  $1 \leq i \leq R$ , we have  $u_i \leq \beta'$ . Hence  $|\text{Pos}(u)| \leq (\frac{2c_0}{t})R = \frac{R^\gamma}{2}$ . Further, since all the constraints  $\mathcal{P}_2$  are  $\beta'$ -satisfied, we know that

$$\left| \sum_{i \in \pi_e^{-1}(j)} u_i - v_j \right| \leq \beta'$$

Thus for every label,  $j \in \text{Pos}(v)$ , there is at least one label  $i \in \text{Pos}(u)$  such that  $\pi_e(i) = j$ . For every vertex  $w \in U \cup V$ , assign a label chosen uniformly at random from  $\text{Pos}(w)$ . For any *good* edge  $e = (u, v)$ , the probability that the constraint  $\pi_e$  is satisfied is at least  $\frac{1}{|\text{Pos}(u)|} \geq \frac{2}{R^\gamma}$ . Since at least half of the edges are *good*, this shows that there is an assignment to the label cover instance that satisfies at least a fraction  $1/R^\gamma$  of the edges.  $\square$

## 5 Linear equations over Rationals

**Theorem 5.1.** For all  $\varepsilon, \delta > 0$ , the problem  $\text{LINEQ-MA}(1 - \varepsilon, \delta)$  is NP-hard.

*Proof.* Given a label cover instance  $\Pi$ , the reduction outlined in Theorem 4.1 is applied with  $\varepsilon_1 = \varepsilon$ ,  $\delta_1 = \frac{\delta}{2}$  to obtain a set of equation tuples  $\mathcal{T}$ . From  $\mathcal{T}$ , a set of equations over  $\mathbb{Q}$  is obtained as follows:

For each tuple  $T = (\{E_1, \dots, E_n\}, E) \in \mathcal{T}$ , include the following set of equations:

$$E_1 + y \cdot E_2 + \dots + y^{n-1} E_n + y^n (E - 1) = 0$$

for all values of  $y = 1, 2, \dots, t$ , where  $t = \frac{(n+1)}{\delta_1}$ .

**Completeness:** Observe that if  $\Pi$  is satisfiable then the corresponding assignment  $A$  has a scaling factor  $E(A) = 1$ . Further for every equation tuple  $T$  that is satisfied by  $A$ ,  $E_i(A) = 0, 1 \leq i \leq n$ . Hence  $A$  satisfies at least  $1 - \varepsilon$  fraction of the equations.

**Soundness:** Suppose there is an assignment  $A$  that satisfies at least a fraction  $\delta = 2\delta_1$  of the equations. Hence for at least  $\delta_1$  fraction of the tuples, at least  $\delta_1$  fraction of the equations are satisfied. Let us refer to these tuples as *nice*. If a tuple  $T$  is not satisfied by  $A$ , then at most  $\frac{n+1}{t} < \delta_1$  fraction of the equations corresponding to  $T$  can be satisfied. Hence every *nice* tuple  $T$  is satisfied by  $A$ . So the assignment  $A$  satisfies at least a fraction  $\delta_1$  of the tuples, which is a contradiction to Theorem 4.1.  $\square$

The coefficients of variables in the above reduction could be exponential in  $n$  (their binary representation could use polynomially many bits). Using an alternate reduction, we can prove a similar hardness even if all the coefficients are bounded by a constant depending only on  $\varepsilon, \delta$ , and moreover the arity of all the equations (i.e., the number of variables with a nonzero coefficient) is also bounded by a constant. We omit the proof here.

**Theorem 5.2.** *For any constants  $\varepsilon, \delta > 0$ , there exist  $B, b > 0$  such that LINEQ-MA( $1 - \varepsilon, \delta$ ) is NP-hard even on linear systems where each equation has arity at most  $b$  and all coefficients are bounded in absolute value by  $B$ .*

## 6 Verifier II

The main ideas in the construction of the second verifier are described below.

The equation tuple  $T$  that needs to be tested may not be disjoint, i.e., there could be a variable that occurs in more than one equation in  $T$ . This problem can be solved by using multiple copies of each variable, and using different copies for different equations. However, it is important to ensure that the different copies of the variables are consistent. To ensure this the verifier does the following : it has a very large number of copies of each variable in comparison to the number of equations. On all the copies that are not used for equations in  $T$ , the verifier checks pairwise equality. Any given copy of a variable is used to check an equation in  $T$  for only a very small fraction of cases, and for most random choices of Verifier II, the copy of the variable is used for consistency checking. This way most of the copies are ensured to be consistent with each other.

The pairwise consistency checks made between the copies must also satisfy the disjointness property. So the verifier picks a matching at random, and performs pairwise equality checks on the matching. It can be shown that even

if there are a small number of bad copies, they will get detected by the matching with high probability.

If a single equation is unsatisfied in  $T$ , at least  $C$  equations need to be unsatisfied on the output tuple. This is easily ensured by checking each equation in  $T$  on many different copies of the variables. As all the copies are consistent with each other, if one equation is unsatisfied in  $T$  a large number of equations in the output tuple will be unsatisfied.

Let us say the tuple  $T$  consists of equations  $E_1, \dots, E_m$  and a scaling factor  $E$  over variables  $u_1, \dots, u_n$ . Let us denote by  $n_0$  the maximum arity of an equation in  $T$ . We use superscripts to identify different copies of the variables. Thus  $u_i^j$  refers to the variable corresponding to  $j^{\text{th}}$  copy of the variable  $u_i$ . Further for an equation/linear function  $E$ , the notation  $E^j$  refers to the equation  $E$  over the  $j^{\text{th}}$  copies of variables  $V(E)$ . By the notation  $M_i(j, k)$ , we refer to the following pairwise equality check:

$$M_i(j, k) : \quad u_i^j - u_i^k = 0.$$

Let  $M, P$  be parameters (even integers) whose values will be chosen later. The set of variables used by Verifier II consists of  $M$  copies for variables not in  $V(E)$ , and  $M + 1$  copies of variables in  $V(E)$

We now define family of pseudo random permutations that we will use in Verifier II.

**Definition 6.1.** *Two distributions  $D_1, D_2$  over a finite set  $\Omega$  are said to be  $\eta$ -close to each other if the variation distance  $\|D_1 - D_2\| = \frac{1}{2} \sum_{\omega \in \Omega} |D_1(\omega) - D_2(\omega)|$  is at most  $\eta$ .*

**Definition 6.2.** *A family of permutations  $\Pi$  (can have repetitions) of  $[1 \dots M]$  is said to be  $k$ -wise  $\eta$ -dependent if for every  $k$ -tuple of distinct elements  $(x_1, \dots, x_k) \in [1 \dots M]$ , the distribution  $(f(x_1), f(x_2), \dots, f(x_k))$  for  $f \in \Pi$  chosen uniformly at random is  $\eta$ -close to the uniform distribution on  $k$ -tuples.*

Let  $\Pi$  denote a set of 4-wise  $\eta$ -dependent permutations of  $\{1, \dots, M\}$ . Explicit constructions of such families of permutations of size polynomial in  $M$  (specifically  $\left(\frac{M}{\eta}\right)^{O(1)}$ ) are known, see [19, 15].

**Theorem 6.3.** *For all  $\varepsilon_2, \delta_2 > 0$  and a positive integer  $C$  there exists constants  $P, \eta$  such that: Given a set of equation tuples  $\mathcal{T}$  of which each tuple is of constant arity( $n_0$ ) and has the same scaling factor  $E$ , the following holds with sufficiently large choice of  $M$ .*

- *If an assignment  $A$ , satisfies a fraction  $1 - \varepsilon_2$  of the tuples  $T \in \mathcal{T}$  then there exists an assignment  $A'$  which satisfies a fraction  $1 - \varepsilon_2$  of the tuples output by the verifier.*
- *If no assignment  $\beta'$ -satisfies a fraction  $\frac{\delta_2}{2}$  of the tuples  $T \in \mathcal{T}$ , then no assignment  $A'$  is  $C$ -close to  $\beta = \frac{\beta'}{9n_0}$ -satisfying a fraction  $\delta_2$  of the output tuples.*

- Pick an equation tuple  $T = (\{E_1, E_2, \dots, E_r\}, E) \in \mathcal{T}$  uniformly at random.
- Pick a number  $k$  uniformly at random from  $\{1, \dots, M+1\}$ . Choose  $E^k$  as the scaling factor. Re-number the remaining  $M$  copies of  $V(E)$ , with  $\{1, \dots, M\}$ .
- Choose a permutation  $\pi$  uniformly at random from the set  $\Pi$  of 4-wise  $\eta$ -dependent permutations. Construct sets of equations  $\mathcal{P}$  and  $\mathcal{M}$  as follows:
 
$$\mathcal{P} = \{E_\ell^{\pi(j)} \mid 1 \leq \ell \leq r, (P-1)\ell + 1 \leq j \leq P\ell\}$$

$$\mathcal{M} = \{M_i(\pi(j), \pi(j+1)) \mid u_i^{\pi(j)} \notin V(\mathcal{P}), j \text{ odd}\}$$
- Output the tuple  $(\mathcal{P} \cup \mathcal{M}, E^k)$ .

*Proof.* The completeness proof is clear, since an assignment  $A'$  consisting of several copies of  $A$  satisfies the exact same tuples that  $A$  satisfies.

Suppose an assignment  $A'$  is  $C$ -close to  $\beta$ -satisfying  $\delta_2$ -fraction of the output tuples. Then for at least a fraction  $\frac{\delta_2}{2}$  choices of input tuple  $T \in \mathcal{T}$ , at least a fraction  $\frac{\delta_2}{2}$  of the output tuples are  $C$ -close to being  $\beta$ -satisfied by  $A'$ . Let us call these input tuples  $T$  to be *good*. For a good tuple  $T$ , there are at least  $\frac{\delta_2}{4}$  fraction of choices of  $k$  for which with probability more than  $\frac{\delta_2}{4}$ , the output tuple is  $C$ -close to being  $\beta$ -satisfied (by  $A'$ ). These values of  $k$  (and the associated copy of the scaling factor  $E^k$ ) are said to be *nice* with respect to  $T$ .

**Lemma 6.4.** *Let  $E^k$  be a nice scaling factor of  $T$ . Then, for every equation  $E_\ell \in T$ , there exist at least  $P-C$  values of  $j$  for which  $|E_\ell^j(A')| \leq \beta|E^k(A')|$*

*Proof.* Since  $E^k$  is a nice scaling factor, for at least one permutation  $\pi \in \Pi$ , the assignment  $A'$  is  $C$ -close to  $\beta$ -satisfying the generated tuple. Since each equation  $E_\ell$  is checked on  $P$  different copies, at least  $P-C$  of the copies must be  $\beta$ -satisfied by  $A'$ .  $\square$

**Lemma 6.5.** *For sufficiently large choice of the parameter  $C_0$ , the following holds: Let  $E^k$  be a scaling factor that is nice with respect to some good tuple  $T$ . For every variable  $u_i$  that occurs in the equations of  $T$  (including  $E$ ), all but  $C_0$  of copies of  $u_i$  are  $2\beta|E^k(A')|$  close to each other, i.e.,  $|A'(u_i^{j_1}) - A'(u_i^{j_2})| \leq 2\beta|E^k(A')|$  for all but  $C_0$  values of  $1 \leq j_1, j_2 \leq M$ .*

*Proof.* As  $E^k$  is a nice scaling factor w.r.t.  $T$ , for at least a fraction  $\frac{\delta_2}{4}$  choices of  $\pi \in \Pi$  the assignment  $A'$  is  $C$ -close to  $\beta$ -satisfying the output tuple  $\mathcal{P} \cup \mathcal{M}$ . In particular, this

means that with probability at least  $\frac{\delta_2}{4}$ , at most  $C$  of the consistency checks in  $\mathcal{M}$  fail to be  $\beta$ -satisfied.

Define a copy  $u_i^j$  to be *far* from  $u_i^{j_1}$  if  $|A'(u_i^j) - A'(u_i^{j_1})| > \beta|E^k(A')|$ . We call a copy  $u_i^j$  to be *bad*, if it is far from at least  $M/2$  other copies. Suppose there are more than  $C_0$  bad copies of the variable  $u_i$ . Without loss of generality we can assume that the first  $C_0$  copies  $\{u_i^1, u_i^2, \dots, u_i^{C_0}\}$  are bad. We will prove below that, for large enough  $C_0, M$ , with high probability over the choice of  $\pi \in \Pi$ , at least  $2C+1$  of these bad copies will be involved in checks in  $\mathcal{M}$  that are not  $\beta$ -satisfied. This will in turn imply that more than  $C$  of the checks in  $\mathcal{M}$  are not  $\beta$ -satisfied.

For a uniformly random permutation, the probability that (the index of) a fixed bad copy is the image of a fixed  $j \in \{1, \dots, M\}$  is  $\frac{1}{M}$ . Hence the probability that a fixed bad copy is used for an equation in  $\mathcal{P}$  is at most  $\frac{Pr}{M}$ . Since  $\Pi$  is  $\eta$  independent, the probability that one of the  $C_0$  bad copies  $u_i^j, 1 \leq j \leq C_0$ , is used for some equation in  $\mathcal{P}$  is at most  $C_0(\frac{Pr}{M} + \eta)$ . So, except with this probability, all bad copies are assigned to consistency checks in  $\mathcal{M}$ .

A bad copy  $u_i^j, 1 \leq j \leq C_0$  fails to be  $\beta$ -satisfied by a check in  $\mathcal{M}$  whenever a far copy is mapped next to it. Let  $Z_j, 1 \leq j \leq C_0$  be the 0, 1 random variable indicating the event that this happens for copy  $u_i^j$ .

Using the 4-wise  $\eta$ -independence of the family of permutations  $\Pi$ , the following can be shown:

- For each  $j, 1 \leq j \leq C_0$  we have

$$\mathbf{E}[Z_j] = \Pr_{\pi \in \Pi}[Z_j = 1] \geq \frac{1}{3}$$

- For all  $1 \leq j_1 < j_2 \leq C_0$

$$\mathbf{E}[Z_{j_1} Z_{j_2}] \leq \mathbf{E}[Z_{j_1}]\mathbf{E}[Z_{j_2}] + \frac{3}{M-3} + 3\eta$$

Define the random variable  $X$  to be  $\sum_{i=1}^{C_0} Z_i$ . Then the expectation  $\mathbf{E}[X]$  and variance  $\sigma^2$  are given by

$$\mathbf{E}[X] = \sum_{i=1}^{C_0} \mathbf{E}[Z_i] \geq \frac{C_0}{3}$$

$$\begin{aligned} \sigma^2 &= \sum_{j_1=1}^{C_0} (\mathbf{E}[Z_{j_1}^2] - (\mathbf{E}[Z_{j_1}])^2) \\ &\quad + \sum_{j_1=1}^{C_0} \sum_{j_2 \neq j_1}^{C_0} (\mathbf{E}[Z_{j_1} Z_{j_2}] - \mathbf{E}[Z_{j_1}]\mathbf{E}[Z_{j_2}]) \end{aligned}$$

$$\sigma^2 \leq C_0 + 2 \binom{C_0}{2} \left( \frac{3}{M-3} + 3\eta \right)$$

Therefore  $\sigma^2 < 2C_0$ , for  $M, \frac{1}{\eta}$  sufficiently large compared to  $C_0$ . Using Chebyshev's inequality, it follows that

$$\Pr[X \leq 2C] \leq \frac{2C_0}{\left(\frac{C_0}{3} - 2C\right)^2}$$

Putting these facts together, it follows that the probability over the choice of  $\pi \in \Pi$  (once a nice value of  $k$  is picked) that at most  $C$  of the consistency checks in  $\mathcal{M}$  fail to be  $\beta$ -satisfied is at most

$$C_0 \left( \frac{Pr}{M} + \eta \right) + \frac{2C_0}{\left(\frac{C_0}{2} - 2C\right)^2} < \frac{\delta_2}{4}$$

provided  $M > \frac{40PrC_0}{\delta_2}$  and  $C_0, \frac{1}{\eta}$  are chosen to be sufficiently large constants compared to  $C, 1/\delta_2$ . This contradicts the niceness of the scaling factor  $E^k$ .

It must thus be the case that at most  $C_0$  copies of the variable  $u_i$  are bad. Now if neither of the copies  $u_i^{j_1}$  and  $u_i^{j_2}$  are bad, then both  $A'(u_i^{j_1})$  and  $A'(u_i^{j_2})$  are within  $\beta|E^k(A')|$  of the the value assigned by  $A'$  to more than half the copies of  $u_i$ . This implies that they must themselves be within  $2\beta|E^k(A')|$  of each other. Thus all but  $C_0$  copies of  $u_i$  are  $2\beta|E^k(A')|$  close to each other.  $\square$

Returning to the proof of Theorem 6.3, fix  $T^*$  to be an arbitrary good tuple. Define  $k_0$  to be its nice value for which the corresponding scaling factor  $E^{k_0}(A')$  has the smallest absolute value. (Note that  $E^k(A) > 0$  for every scaling factor  $E^k$  that is nice with respect to  $T^*$ , hence  $E^{k_0}(A') > 0$ .) From Lemma 6.5, we know that all but  $C_0$  of the copies of every variable are  $2\beta|E^{k_0}(A')|$  close to each other. Delete all the bad copies (at most  $C_0$ ) of each variable. Further, delete all the variables in  $V(E^{k_0})$ . Now define an assignment  $A$  as follows: The value of  $A(u_i)$  is the average of all the copies of  $u_i$  that have survived the deletion. We claim that the assignment  $A$   $\beta'$ -satisfies all the good tuples  $T' \in \mathcal{T}$ .

Observe that the arity of scaling factor  $E$  is at most  $n_0$ , and at most  $C_0 + 1$  copies of each variable are deleted. Since there are at least  $\frac{\delta_2}{4}M$  nice scaling factors and  $\frac{\delta_2}{4}M > n_0(C_0 + 1)$ , there exists a nice scaling factor  $E^{k_1}$  of  $T^*$  such that no variable of  $V(E^{k_1})$  is deleted. Further by definition of  $k_0$ ,  $|E^{k_1}(A')| \geq |E^{k_0}(A')|$ .

From Lemma 6.5, for the average assignment  $A$  and any undeleted variable  $u_i^j$  occurring in an equation of  $T^*$ , we have

$$|A(u_i) - u_i^j| \leq 2\beta|E^{k_0}(A')| \leq 2\beta|E^{k_1}(A')|. \quad (2)$$

Using the above for the variables in  $V(E^{k_1})$ , we get

$$(1 - 2\beta n_0)|E^{k_1}(A')| \leq |E(A)|$$

Substituting back in (2), we get

$$|A(u_i) - u_i^j| \leq \frac{2\beta}{(1 - 2\beta n_0)}|E(A)| \leq 4\beta|E(A)| \quad (3)$$

Consider any good tuple  $T' \in \mathcal{T}$ . By the same argument used for  $T^*$ , it can be shown that there exists a scaling factor  $E^{j_0}$  that is nice with respect to  $T'$  and none of whose

variables have been deleted. Using the analog of (3) for variables in  $V(E^{j_0})$ , we have

$$|E^{j_0}(A')| \leq |E(A)| + 4\beta \cdot n_0|E(A)| \quad (4)$$

Using Lemma 6.4, and the fact  $P - C > n_0C_0$ , we can conclude for every equation  $E_\ell \in T'$ , there exists  $j_1$  such that  $|E_\ell^{j_1}(A')| \leq \beta|E^{j_0}(A')|$ , and no variable of  $V(E_\ell^{j_1})$  is deleted. Similar to (4) we get

$$|E_\ell(A)| \leq |E_\ell^{j_1}(A')| + 4\beta \cdot n_0|E(A)|$$

Therefore,

$$\begin{aligned} |E_\ell(A)| &\leq (\beta + 4\beta^2 n_0 + 4\beta n_0)|E(A)| \\ &\leq 9\beta n_0|E(A)| = \beta'|E(A)| \end{aligned}$$

implying that the assignment  $A$   $\beta'$ -satisfies the tuple  $T'$ . Hence the assignment  $A$   $\beta'$ -satisfies all the good tuples. Recalling that at least a fraction  $\delta_2/2$  of the tuples are good, the result of Theorem 6.3 follows.  $\square$

## 7 Verifier III

Given a equation tuple  $T = (\{E_1, \dots, E_n\}; E)$ , Verifier III checks whether the assignment  $A$  satisfies  $T$  or is not even  $C$ -close to  $\beta$ -satisfying  $T$ . Towards this, we define the following notation : For a tuple of equations  $\mathcal{E} = (E_1, \dots, E_n)$ , and a vector  $v \in \{-1, 1\}^n$ , define  $\mathcal{E} \cdot v = \sum_{i=1}^n v_i E_i$ .

Let  $V_i$  for an integer  $i$ , denote a 4-wise independent subset of  $\{-1, 1\}^i$ . Polynomial size constructions of such sets are well known, see for example [2, Chap. 15]. The details of the verifier are described below.

- Partition the set of equations  $\{E_1, \dots, E_n\}$  using  $n$  random variables that are  $C$ -wise independent and take values  $\{1, \dots, m\}$ . Let us say the partitions are  $\mathcal{E}_i, 1 \leq i \leq m$ .
- For each partition  $\mathcal{E}_i$ , pick a random vector,  $v_i \in V_{n_i}$  where  $n_i = |\mathcal{E}_i|$ . Compute linear functions  $B_i, 1 \leq i \leq m$

$$B_i = \mathcal{E}_i \cdot v_i$$

Construct  $B = (B_1, B_2, \dots, B_m)$

- Pick a vector  $w$  uniformly at random from  $\{-1, 1\}^m$ .
- With probability  $\frac{1}{2}$ , check one of the following two inequalities:

$$B \cdot w + E \geq \theta \quad (5)$$

$$B \cdot w - E < \theta \quad (6)$$

Accept if the check is satisfied, else Reject.

Polynomial size spaces for  $C$ -wise independent variables taking values  $\{1, \dots, m\}$ , can be obtained using BCH codes with alphabet size  $m$ , and minimum distance  $C + 1$ .

**Theorem 7.1.** *For every  $\beta, \delta_3 > 0$  there exist constants  $C = C(\beta, \delta_3), m$  such that the following holds: Given the equation tuple  $T = (\{E_1, \dots, E_n\}, E)$  and an assignment  $A$ ,*

- *If the assignment  $A$  satisfies  $T$ , then with  $\theta = 0$ , the verifier accepts with probability 1.*
- *If the assignment  $A$  is  $C$ -far from  $\beta$ -satisfying the tuple  $T$ , then irrespective of the value of  $\theta$ , the verifier accepts with probability less than  $\frac{1}{2} + \frac{\delta_3}{2}$ .*

*Proof.* For an assignment  $A$  that satisfies the tuple  $T$ , we have  $E_j(A) = 0, 1 \leq j \leq n$ , and  $E(A) > 0$ . Hence for all the random choices,  $B = 0$ , and  $E > 0$ . Therefore, with the choice  $\theta = 0$ , all the checks made by the verifier succeed. (In fact, the  $\geq$  conditions hold with a strict inequality.)

Suppose the assignment  $A$  is  $C$ -far from  $\beta$ -satisfying the tuple  $T$ . If  $E(A) \leq 0$ , then clearly at most one of these two inequalities 5 can be satisfied, and the proof is complete. Hence, we assume  $E(A) > 0$ .

There are at least  $C$  values  $\{E_j(A) | 1 \leq j \leq n\}$  that have absolute value greater than  $\beta|E(A)|$ . Let us refer to these  $E_j$  as *large*. The probability that one of the partitions  $\mathcal{E}_i$  contains less than  $B_0 = \frac{2}{\beta^2}$  *large* values is at most  $m \binom{C}{B_0} (1 - \frac{1}{m})^{C-B_0}$ . From Lemma 7.2, for a partition  $\mathcal{E}_i$  that has at least  $B_0$  *large* values,

$$\Pr[|B_i(A)| > |E(A)|] \geq \frac{1}{12}$$

Assuming, that all the partitions have at least  $B_0$  *large* values, we bound the probability that less than  $\frac{m}{24}$  partitions have  $|B_i(A)| > |E(A)|$ . Towards this, we use the Chernoff bounds, to obtain

$$\Pr[\{i : |B_i(A)| > |E(A)|\} < \frac{m}{24}] \leq e^{-\frac{m}{96}}$$

Consider the case in which there are at least  $m_0 = \frac{m}{24}$  partitions with  $|B_i(A)| > |E(A)|$ . Let  $Z$  denote the event  $B \cdot w - \theta \in [-E(A), E(A)]$ . From Lemma 7.3 we can conclude

$$\Pr[Z] \leq \frac{\binom{m_0}{m_0/2}}{2^{m_0-1}}$$

Overall we have,

$$\Pr[Z] \leq m \binom{C}{B_0} \left(1 - \frac{1}{m}\right)^{C-B_0} + e^{-\frac{m}{96}} + \frac{\binom{m_0}{m_0/2}}{2^{m_0-1}}$$

The value of  $B_0 = \frac{2}{\beta^2}$  is fixed, so for large enough values of  $C, m$  with  $C > m$  the above probability is less than  $\delta_3$ . Observe that if  $B \cdot w - \theta \notin [-E(A), +E(A)]$ , at most one

of the two checks performed by the verifier can be satisfied. Hence the probability of acceptance of the verifier is less than  $\frac{1}{2} + \frac{\delta_3}{2}$ .  $\square$

**Lemma 7.2.** *For all  $\beta > 0$ , and a constant  $B_0 \geq \frac{2}{\beta^2}$ , if  $V \subseteq \{-1, 1\}^n$  is a 4-wise independent space of vectors then for any  $a \in \mathbb{R}^n$  with at least  $B_0$  of its components greater than  $\beta$  in absolute value,*

$$\Pr[|a \cdot v| > 1] \geq \frac{1}{12}$$

where the probability is over random choice of  $v \in V$ .

*Proof.* Define a random variable  $x = |a \cdot v|^2$  for  $v$  chosen uniformly at random from  $V$ . Then it can be shown that,

$$\mathbf{E}[x] = \|a\|_2^2, \quad \mathbf{E}[x^2] = 3\|a\|_2^4 - 2\|a\|_4^4 < 3\|a\|_2^4$$

Since at least  $B_0$  components of  $a$  are larger than  $\beta$ , we have  $\|a\|_2^2 > B_0\beta^2 \geq 2$ . Therefore, if  $\Pr[|a \cdot v| > 1] = \alpha < \frac{1}{12}$ , then

$$\mathbf{E}[x|x > 1] \geq \frac{1}{\alpha} (\|a\|_2^2 - (1 - \alpha) \cdot 1) > \frac{1}{2\alpha} \|a\|_2^2$$

Using the Cauchy-Schwartz inequality, we know

$$\mathbf{E}[x^2|x > 1] \geq (\mathbf{E}[x|x > 1])^2 > \frac{1}{4\alpha^2} \|a\|_2^4$$

Therefore, we get

$$\mathbf{E}[x^2] \geq \mathbf{E}[x^2|x > 1] \Pr[x > 1] > \frac{1}{4\alpha} \|a\|_2^4 > 3\|a\|_2^4$$

which is a contradiction.  $\square$

**Lemma 7.3.** *For every vector  $a \in \mathbb{R}^m$  with at least  $K$  of its components  $> 1$  in absolute value and a number  $\theta \in \mathbb{R}$ ,*

$$\Pr[\theta - 1 \leq a \cdot v \leq \theta + 1] \leq \frac{\binom{K}{K/2}}{2^{K-1}}$$

where the probability is over random choice of  $v \in \{-1, 1\}^m$ .

*Proof.* Without loss of generality, we can assume that  $a_i > 1$  for  $1 \leq i \leq K$ . For a vector  $v \in \{-1, 1\}^m$ , we write  $v = v_{|K} \circ v_{|m-K}$  where  $v_{|K} \in \{-1, 1\}^K, v_{|m-K} \in \{-1, 1\}^{m-K}$  and  $\circ$  denotes the concatenation of the two vectors. Denote by  $\mathbf{-1}$ , and  $\mathbf{1}$  the  $K$  dimensional vectors consisting of all  $-1$ s and all  $1$ s respectively. Consider a path  $\mathcal{P}$  on the hypercube, starting at  $u_0 = \mathbf{-1} \circ v_{|m-K}$  and reaching  $u_K = \mathbf{1} \circ v_{|m-K}$  by changing one variable from  $-1$  to  $1$  at each step. If  $u_i, u_{i+1}$  are the  $i^{\text{th}}$  and  $(i+1)^{\text{st}}$  nodes on the path  $\mathcal{P}$ , then we know  $a \cdot u_{i+1} - a \cdot u_i = a_i > 1$

Therefore, at most two points on the path  $\mathcal{P}$  can belong to an interval  $[\theta - 1, \theta + 1]$ . In total there are  $K!$  paths  $\mathcal{P}$  from  $u_0$  to  $u_K$ . Further any vector  $v'$  of the form  $v' = v'_{|K} \circ v'_{|m-K}$  is present on at least  $\frac{K}{2}! \frac{K}{2}!$  different paths. Hence, we can conclude  $\Pr[\theta - 1 \leq a \cdot v \leq \theta + 1] \leq \frac{2^{\binom{K}{K/2}}}{2^{K-1}}$ .  $\square$

## 8 Hardness of HS-MA: Putting the Verifiers Together

**Theorem 8.1 (Main Result).** *For all  $\varepsilon, \delta > 0$ , the problem HS-MA( $1 - \varepsilon, \frac{1}{2} + \delta$ ) is NP-hard.*

*Proof.* Given a label cover instance  $\Gamma$ , we use Verifier I with parameters  $\delta_1 = \frac{\delta}{4}, \varepsilon_1 = \varepsilon$  to obtain a set of equation tuples  $\mathcal{T}$ . Let  $R = R(\varepsilon_1, \delta_1)$  denote the parameter obtained in Theorem 4.1. Note that the maximum arity of the equations in the tuples is  $2R$ . Using the set of equation tuples  $\mathcal{T}$  as input, Verifier II with parameters  $\varepsilon_2 = \varepsilon_1, \delta_2 = \frac{\delta}{2}, \beta' = \frac{1}{R^3}$  and arity  $n_0 = 2R$  generates a set of equation tuples  $\mathcal{T}'$ . Apply Theorem 7.1 with  $\delta_3 = \delta, \beta = \frac{1}{18R^4}$  to check one of the equation tuples  $T \in \mathcal{T}'$ .

**Completeness:** If the label cover instance  $\Gamma$  is satisfiable, Verifier I outputs a set of tuples, such that there is an assignment satisfying  $1 - \varepsilon_1 = 1 - \varepsilon$  of the output tuples. Hence by applying Theorems 6.3, 7.1, it is clear that there is an assignment  $A$ , that satisfies at least  $1 - \varepsilon$  of the inequalities.

**Soundness:** Suppose there is an assignment  $A$ , which satisfies  $\frac{1}{2} + \delta$  fraction of the inequalities, then for at least  $\frac{\delta}{2}$  fraction of the tuples  $T \in \mathcal{T}'$ , Verifier III accepts with probability at least  $\frac{1}{2} + \frac{\delta}{2}$ . Therefore  $A$  is  $C$ -close to  $\beta$ -satisfying at least  $\frac{\delta}{2} = \delta_2$ -fraction of the tuples  $T \in \mathcal{T}'$ . Using Theorem 6.3, it is clear that there exists an assignment  $A'$  which  $\beta'$ -satisfies at least a fraction  $\frac{\delta_2}{2} = \frac{\delta}{4} = \delta_1$  fraction of tuples  $T \in \mathcal{T}$ . Hence by Theorem 4.1, the label cover instance  $\Gamma$  has an assignment that satisfies at least a fraction  $\frac{1}{R^\gamma}$  of its edges.

The number of random bits used by the Verifier I is given by  $O(R^{1-\gamma} \log n)$ . In Verifier II a total of  $R^\gamma \log n + \log M + C_1 \log M + \log \frac{1}{\eta} = O(\log n)$  random bits are needed. Verifier III uses at most  $(C-1) \log n + 2 \sum \log n_i + m = O(\log n)$  random bits. Hence the entire reduction from LABELCOVER to HS-MA is a polynomial time reduction.  $\square$

By choosing the parameters of the above reduction appropriately, the following stronger hardness result can be shown

**Theorem 8.2.** *There exists  $\gamma > 0$  such that for all constants  $c > 0$ , the problem HS-MA( $1 - \frac{1}{2^{(\log n)^\gamma}}, \frac{1}{2} + \frac{1}{(\log n)^c}$ ) is Quasi-NP-hard.*

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