Homework 1: Counting

For each problem, remember you must briefly explain/justify how you obtained your answer, as correct answers without an explanation will not receive full credit. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide.

In general, your goal in an explanation is to write enough that a student from class who has attended lecture, but not read the problem yet, could understand your approach, verify your reasoning, and believe your answer is correct. While we do not usually need to see arithmetic, you must include enough work that in principle one could rederive your answer with only a scientific calculator.

Unless a problem states otherwise, you should leave your answer in terms of factorials, combinations, etc., for instance 26^7 or $26!/7!$ or $26 \cdot \binom{26}{7}$ are all good forms for final answers.

Instructions as to how to upload your solutions to Gradescope are on the course web page.

Remember that you must tag your written problems on Gradescope.

Submission: You must upload a **pdf** of your written solutions to Gradescope under "HW 1 [Written]". (Instructions as to how to upload your solutions to Gradescope are on the course web page.) The use of LaTeX is *highly recommended*. (Note that if you want to hand-write your solutions, you'll need to scan them. We will take off points for hand-written solutions that are difficult to read due to poor handwriting and neatness.) Your code will be submitted under "HW 1 [Coding]" as files called `cse312_pset1_vars.py, cse312_pset1_loops.py, cse312_pset1_lists.py, cse312_pset1_functions.py, and cse312_pset1_classes.py.`

LaTeX **practice EdStem lesson**: `https://edstem.org/us/courses/6206/lessons/16640/`

Due Date: This assignment is due at 11:59 PM Thursday July 1 (Seattle time, i.e. GMT-7).
You will submit the written problems as a PDF to Gradescope. Please put each numbered problem on its own page of the pdf (this will make selecting pages easier when you submit), and ensure that your pdfs are oriented correctly (e.g. not upside-down or sideways). The coding problem will also be submitted to Gradescope.

Collaboration: Please read the full collaboration policy. If you work with others (and you should!), you must still write up your solution independently and name all of your collaborators somewhere on your assignment.

1. Softball [15 points]

Nine undergraduate students (5 sophomores and 4 juniors) on CSE's softball team show up for a game.

Note that in softball there are 4 unique infielder positions.

(a) How many ways are there to choose which 4 players will be infielders (for this part, you should just account for who is playing and who is not)?

(b) How many ways are there to assign the 4 infield positions by selecting players from the 9 people who show up (for this part account for both who plays and what position they are playing)?

(c) How many ways are there to choose which 4 players will be infielders if at least one of these players must be a junior (go back to not deciding who plays which position for this part)?

2. Getting from here to there [20 points]

(a) How many paths are there from point (0,0) to (120,180) if every step increments one coordinate and leaves the other unchanged?

(b) How many paths are there from point (0,0) to (120,180) if every step increments one coordinate and leaves the other unchanged and you want the path to go through (100,90)?

(c) How many paths are there from point (0,0) to (120,180) if every step increments one coordinate and leaves the other unchanged and the path **cannot** go through (30, 60) or (100,90)? (Try inclusion-exclusion.)

(d) How many paths are there from point (0,0,0) to (400,20,30) if every step increments one coordinate and leaves the other two unchanged?

3. Sitting around [15 points]

Archer (A), Bilbo (B), Cersei (C), Dante (D), Eowyn (E), Frodo (F), and Gollem (G) are sitting in a row of twelve seats (Note: there are only seven people). Archer and Bilbo are exes, so they cannot sit next to each other. Cersei and Dante are dating, so they must sit next to each other. Eowyn, Frodo, and Gollem are best friends, so they also want to sit next to each other, but Frodo must be in the middle of Eowyn and Gollem (with no spaces between the three). Our goal is to figure out how many ways they can sit in a row. Build up to the answer by answering the following questions: In how many ways can they sit in a row? (*Hint:* We will start by grouping Eowyn, Frodo, and Gollem, as well as Cersei and Dante. Then we will work on placing Archer and Bilbo). Also, you will find the problem easier if you just call them, A,B,C,D, E, F, and G, as we do from now on.

(a) How many ways there are to place the 7 people into the 12 chairs if EFG must sit together in that order and CD must sit together in that order (This is not unlike the rearrangements of SEATTLE that we discussed in lecture (where the empty seats are like the two Ts or two Es.)

(b) How many ways there are to place the 7 people into the 12 chairs if EFG must sit together but E and G can swap positions and CD must sit together in either order?

(c) How many ways there are to place the 7 people into the 12 chairs if EFG must sit together (but E and G can swap positions), CD must sit together in either order and AB must sit together in either order?

(d) How many ways there are to place the 7 people into the 12 chairs if EFG must sit together (but E and G can swap positions), CD must sit together in either order and A and B must not sit next to each other?

4. Binomial Theorem applications [15 points]

For part (a) of this question (as with many others for 312), you could find the numerical answer in a few seconds by asking WolframAlpha (in this case, by asking it to expand the polynomial). We have learning goals associated with this problem that mean we want you to practice solving this problem by hand even though you could easily answer it with computational power.

Remember that you must give an explanation of an answer such that another student would understand the **principles** that go into solving the problem, and such that they could find the answer with a simple calculator (that doesn't have an "expand a polynomial" operation)

You may find it beneficial to verify your answer using WolframAlpha, but you may not use the "show steps" option on WolframAlpha or any similar tool.

(a) What is the coefficient of x^7y^{12} in the expansion of $(3x - y^4)^{10}$?

(b) Use the binomial theorem to prove that

$$\sum_{i=0}^{100} \binom{100}{i} (-6)^{100-i} = 5^{100}$$

5. Combinatorial Identities [16 points]

Prove each of the following identities using a *combinatorial argument* (i.e., an argument that counts two different ways); an algebraic solution will be marked substantially incorrect.

For the purposes of these problems, using commutativity of multiplication and addition (i.e. $ab = ba$, $a + b = b + a$), and distributivity/factoring ($a(b + c) = ab + ac$) are allowed as part of a combinatorial argument. Any other algebra facts (e.g. Pascal's Rule about combinations, the definition of combinations/permutations in terms of factorials, cancelling numbers that appear in numerators and denominators) would make it an algebraic solution not a combinatorial one.

(a) $\sum_{k=0}^{m} \binom{m}{k}\binom{n}{k} = \binom{m+n}{m}$. You may assume that $n \geq m \geq 0$.

Hint: Start with the right hand side and imagine you are choosing a team of m people from a group of people consisting of n Americans and m Canadians.

(b) $\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} = \binom{n}{m}2^{n-m}$. Assume that $n \geq m \geq 0$.

Hint: You'll need a different setup for this problem then the last one (i.e., starting with n Americans and m Canadians isn't a good place to start).

6. Classic [20 points]

The goals of this problem are to

- Practice induction (remember induction? It's back!) so that you don't totally forget it before you write another inductive proof in 421.
- Realize that while induction works for proving combinatorial identities, it usually leads to longer proofs than other methods.

(a) Use Pascal's Rule to show that for any $n \geq 0$

$$\sum_{k=0}^{n} \binom{n}{k} + \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n+1} \binom{n+1}{k}$$

(you do not need to use induction for this part)

(b) Now, show via induction that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ holds for all natural numbers n. You may use part (a) in your inductive step.

(c) We've seen in lecture that this theorem can be proven very quickly via the binomial theorem. There is also a combinatorial proof (counting subsets of a set of size n, on the left hand side we consider the subsets of size k); of the three versions of this proof (binomial theorem, induction, combinatorial) which do you prefer? Why? (Write 1-3 setences; there are not right or wrong answers).

7. Coding [15 points]

(a) Read the HW1 Python Tutorial and Coding Exercises lesson on Edstem and follow the directions to complete 5 coding exercises. Then submit all required files to HW1 [Coding] on Gradescope. You may resubmit to Gradescope as many times as you like. We do not have any hidden tests for this assignment; whatever score you see on Gradescope on your last submission will be your final score.

(b) Read the Edstem lesson on Python's numpy library, after completing the previous part. You do **not** need to complete any coding exercises or submit anything to Gradescope for this part. The exercise that is there is entirely OPTIONAL, and intended only for practice if you want it. Afterward, write what you felt was the most confusing numpy function and/or class to you and why. If nothing is confusing, explain which function and/or class is the most interesting to you. We will grade based on completion and effort rather than correctness, and it's recommended that your answer be no longer than 5 sentences. Include this paragraph with your pdf submission to Gradescope.