Homework 5: Discrete Variable Wrap-up; Intro to Continuous Random Variables

For each problem, remember you must briefly explain/justify how you obtained your answer, as correct answers without an explanation will not receive full credit. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide.

In general, your goal in an explanation is to write enough that a student from class who has attended lecture, but not read the problem yet, could understand your approach, verify your reasoning, and believe your answer is correct. While we do not usually need to see arithmetic, you must include enough work that in principle one could rederive your answer with only a scientific calculator.

Unless a problem states otherwise, you should leave your answer in terms of factorials, combinations, etc., for instance 26^7 or $26!/7!$ or $26 \cdot \binom{26}{7}$ are all good forms for final answers.

Instructions as to how to upload your solutions to gradescope are on the course web page.

Remember that you must tag your written problems on Gradescope.

Submission: You must upload a **pdf** of your written solutions to Gradescope under "HW 5". The use of LaTeX is *highly recommended*. (Note that if you want to hand-write your solutions, you'll need to scan them. We will take off points for hand-written solutions that are difficult to read due to poor handwriting and neatness.)

Please put each numbered problem on its own page of the pdf (this will make selecting pages easier when you submit), and ensure that your pdfs are oriented correctly (e.g. not upside-down or sideways).

Due Date: This assignment is due at 11:59 PM Thursday July 29 (Seattle time, i.e. GMT-7).

Collaboration: Please read the full collaboration policy. If you work with others (and you should!), you must still write up your solution independently and name all of your collaborators somewhere on your assignment.

1. Dice Try! [15 points]

You are playing a game that uses a fair 14-sided die whose faces are numbered 1, 2, ..., 14. The value of a roll is the number showing on the top of the die when it comes to rest. Give all answers as simplified fractions.

(a) Let X be the value of one roll of the die. Compute $\mathbb{E}[X]$ and $\text{Var}(X)$. For this problem, you can either calculate from first principles (e.g. definitions and theorems) or describe which variable from the zoo is appropriate and what values you get as a result.

(b) Let Y be the sum of the values of 5 independent rolls of the die. Compute $\mathbb{E}[Y]$ and $\text{Var}(Y)$. Use independence, and state precisely where in your computation you are using it.

(c) Let Z be the **average** of the values of 5 independent rolls of the die. Compute $\mathbb{E}[Z]$ and $\text{Var}(Z)$. Use independence, and state precisely where in your computation you are using it.

2. If you can dodge a wrench... [8 points]

Your dodgeball team has reached the championship game. You start the game with an equal number of players.

One of your team members will throw a ball at an opponent, with probability p, they hit an opponent and your opponents lose a teammate. With probability q, your oponents catch that ball: you lose a teammate and they gain a teammate. With probability $1 - p - q$, the opponent dodges and nothing changes. You then resume with a team member throwing a ball, according to the same rules.

These throws are all independent of each other and the number of teammates on each team. This continues until:

- Your team has at least two more players than your opponents team. At that point you change to your patented telescoping-triple-team strategy, which guarantees you will win.
- Or, your opponents have at least two more players than your team. At that point, your opponents execute the dangerous-double-double-team play, which guarantees they will win.

Let x be the probability that your team wins, starting from equal number of players, y be the probability that your team wins, starting with one more player, and z be the probabiilty that your team wins, starting with one fewer player.

(a) Write a system of equations, involving variables x, y, z and (unknown) constants p, q that would let you find x, y, z if you knew p, q.

(b) Suppose $p = 1/2, q = 1/3$. What are the values of x, y, z? (You do not have to show work for this part, using wolfram alpha or another computer algebra system is fine).

3. Volleyball [10 points]

A set in volleyball ends when a team has:

- At least 25 points and
- at least 2 more points than its opponent.

So, for example, a set at 25-24 is not over (no one has a two point lead), but a set at 27-25 is over.

Suppose a set is tied 24-24, and each point is won by your team (independently) with probability p. What is the expected number of points played before one team or the other wins the set. (Hint: You could calculate this with an infinite sum, in which case you may use wolframalpha to find a closed form. But a clever use of a vaiable from the zoo will save you quite a bit of work, and make it so an infinite summation is unnecessary.)

4. Rap Battles [10 points]

After using the dating app in HW3, you've decided to try a new strategy to secure a date. You take your favorite 256 suitors and craft a single-elimination bracket tournament where every rap battle occurs between pairs of individuals and the winner in the current rap battle plays some other winner in the next round (and the loser is eliminated). Your friend wants to predict the winners, but knowing nothing about the suitors, so they flip a coin to determine who they will predict to be the winner for each of the 255 rap battles (since the first round has 128 rap battles, the second has 64 rap battles,... and the 8th has 1 rap battle).[1] To thank your friend for their pseudo-psychic efforts you will pay your friend 1 dollar for every correctly predicted winner in the first round, 2 dollars for every correctly predicted winner in the second round, ... and 128 dollars for correctly predicting the final winner (i.e. 2^{i-1} dollars per correct prediction in round i). Determine the expected number of dollars your friend will earn.

5. PDF and CDF [10 points]

For this exercise, give exact answers involving simplified fractions as necessary. If you need to evaluate a definite integral, be sure to list the integral, the antiderivative, and what you get when plugging in the limits of integration as part of your work.

Consider the following PDF:

$$f_X(x) = \begin{cases} c(1 - 6x^2) & \text{if } -\frac{1}{4} \le x \le \frac{1}{4} \\ 0 & \text{otherwise} \end{cases}$$

[1]Note that they are predicting the entire bracket before the first rap battle takes place. So to predict the winner of a second round rap battle, they are choosing between two winners they predicted in the prior round.

Where c is a constant.

(a) What value of c makes the PDF valid?

(b) What is $F_X(k)$?