Please download the activity slide for today! :)

Conditional Probability

CSE 312 Summer 21
Lecture 5

Announcements

Problem Set 1 is due tomorrow at 11:59 pm.

You can take up to 2 late days on an assignment.

Please list your collaborators in the assignment submission.

Today

This Far

Counting

Intro to Discrete Probability

Today

Some more examples

Conditional Probability

Bayes' Rule

Another Example

52 cards
13 values
4 suits

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space Top 2 cards. $\underline{52}\ \underline{51}$

Probability Measure $P(\omega) = \frac{1}{52 \cdot 51}$

Event all pairs with the same value $|E| = \binom{13}{1}\binom{4}{2}2!$

Probability

$$P(E) = \frac{\binom{13}{1} P(4,2)}{52 \cdot 51} = \frac{\binom{13}{1} \cdot P(4,2)}{P(52,2)} = \binom{13}{1} P(4,2)$$

Another Example

$52!$ $\quad P(\omega) = \frac{1}{52!}$

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: $\{(x, y): x$ and y are different cards $\}$

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52 \cdot 51}$

Event: all pairs with equal values

$|E| = \binom{13}{1}\binom{4}{2} 2! \quad 50!$

Probability: $\frac{13 \cdot P(4,2)}{52 \cdot 51}$ ← $P(E) = \frac{\binom{13}{1} P(4,2) \cdot 50!}{52!}$

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: Set of all orderings of all 52 cards

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52!}$

Event: all lists that start with two cards of the same value

Probability: $\frac{13 \cdot P(4,2) \cdot 50!}{52!}$ $= \frac{13 \cdot P(4,2)}{P(52,2)} \Rightarrow \frac{\binom{13}{1} \cdot \binom{4}{2}}{\binom{52}{2}}$

$52 \cdot 3$

$13 \cdot 4 \cdot 3$

$\binom{4}{2} = \frac{P(4,2)}{2!}$

$\binom{52}{2} = \frac{P(52,2)}{2!}$

Takeaway

There's often information you "don't need" in your sample space.

It won't give you the wrong answer.

But it sometimes makes for extra work/a harder counting problem,

Good indication: you cancelled A LOT of stuff that was common in the numerator and denominator.

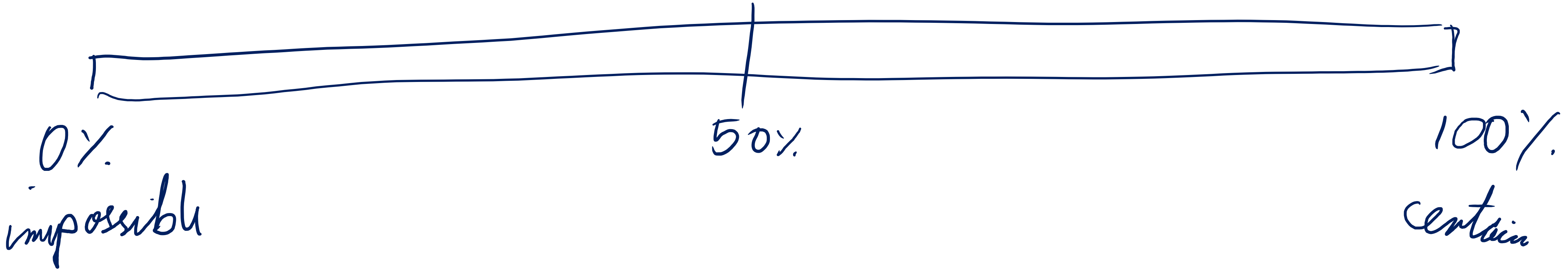
Some Quick Observations

For discrete probability spaces (the kind we've seen so far)

$\mathbb{P}(E) = 0$ if and only if an event can't happen.

$\mathbb{P}(E) = 1$ if and only if an event is guaranteed (every outcome outside E has probability 0).

$$P(E) = 1 - P(\bar{E})$$



Conditional Probabilities

Conditioning

You roll a fair red die and a fair blue die (without letting the dice affect each other).

But they fell off the table and you can't see the results.

I can see the results – I tell you the sum of the two dice is 4.

What's the probability that the red die shows a 5, **conditioned** on knowing the sum is 4?

It's 0.

Without the conditioning (me telling you that the sum in 4) it was $1/6$.

Conditioning

When I told you "the sum of the dice is 4" we restricted the sample space.

The only remaining outcomes are $\{(1,3),(2,2),(3,1)\}$ out of $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$.

Outside the (restricted) sample space, the probability is going to become 0. What about the probabilities inside?

Conditional Probability

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{|A\cap B|}{|\Omega|}}{\frac{|B|}{|\Omega|}}$$

Conditional Probability

For an event B, with $\mathbb{P}(B) > 0$,
the "Probability of A conditioned on B" is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{|A\cap B|}{|B|}$$

Just like with the formal definition of probability, this is pretty abstract.
It does accurately reflect what happens in the real world.

If $\mathbb{P}(B) = 0$, we can't condition on it (it can't happen! There's no point in defining probabilities where we know B has happened) – $\mathbb{P}(A|B)$ is **undefined** when $\mathbb{P}(B) = 0$.

Conditioning…

Let A be "the red die is 5"
Let B be "the sum is 4"
Let C be "the blue die is 3"

$P(B|C) = 1/6$

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
D1=5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
D1=6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Conditioning...

Let A be "the red die is 5"
Let B be "the sum is 4"
Let C be "the blue die is 3"

$\mathbb{P}(A|B)$

$\mathbb{P}(A \cap B) = \mathbb{P}(\emptyset) = 0$
$\mathbb{P}(B) = 3/36$

$P(A|B) = \frac{0}{3/36}$ = 0

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
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Conditioning…

Let A be "the red die is 5"
Let B be "the sum is 4"
Let C be "the blue die is 3"

$\mathbb{P}(A|C)$

$\mathbb{P}(A \cap C) = 1/36$
$\mathbb{P}(C) = 6/36$

$$P(A|B) = \frac{1/36}{6/36} = \frac{1}{6}$$

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
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Conditioning Practice

A → Red is 5
B → Sum is 4
C → Blue is 3

E_1 → Red die 6
E_2 → Sum is 9
E_3 → Sum is 7

- Red die 6 conditioned on sum 7 = 1/6
- Red die 6 conditioned on sum 9 1/4 | Flip would be 1/6
- Sum 7 conditioned on red die 6 = 1/6

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
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Fill out the poll everywhere so Kushal knows how long to explain Go to pollev.com/cse312su21

Conditioning Practice

Red die 6 conditioned on sum 7 1/6

Red die 6 conditioned on sum 9 1/4

Sum 7 conditioned on red die 6 1/6

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
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Direction Matters

$\mathbb{P}(A|B)$ and $\mathbb{P}(B|A)$ are different quantities.

$\mathbb{P}($"I'm using an umbrella" **|** "it's raining"$)$ is pretty small [Seattleites don't use an umbrella.]

$\mathbb{P}($"it's raining" **|** "I'm using an umbrella"$)$ is 1 (or close to it); I don't use an umbrella for anything else.

It's a lot like implications – order can matter a lot!

(but there are some A, B where the conditioning doesn't make a difference)

Wonka Bars

Willy Wonka has placed golden tickets on 0.1% of his Wonka Bars.

You want to get a golden ticket. You could buy a 1000-or-so of the bars until you find one, but that's expensive…you've got a better idea!

You have a test – a very precise scale you've bought.

- If the bar you weigh **does** have a golden ticket, the scale will alert you 99.9% of the time.
- If the bar you weigh does not have a golden ticket, the scale will (falsely) alert you only 1% of the time.

If you pick up a bar and it alerts, what is the probability you have a golden ticket?

Willy Wonka

Willy Wonka has placed golden tickets on 0.1% of his Wonka Bars.

If the bar you weigh **does** have a golden ticket, the scale will alert you 99.9% of the time.

If the bar you weigh does not have a golden ticket, the scale will (falsely) alert you only 1% of the time.

If you pick up a bar and it alerts, what is the probability you have a golden ticket?

Which of these is closest to the right answer?

A. 0.1%
B. 10%
C. 50%
D. 90%
E. 99%
F. 99.9%

Fill out the poll everywhere so Kushal knows how long to explain
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Conditioning

Let A be the event you get ALERTED

Let B be the event your bar has a ticket.

What conditional probabilities are each of these?

Willy Wonka has placed golden tickets on 0.1% of his Wonka Bars. $\mathbb{P}(B)$

If the bar you weigh **does** have a golden ticket, the scale will alert you 99.9% of the time. $\mathbb{P}(A|B)$

If the bar you weigh does not have a golden ticket, the scale will (correctly) not alert you 99% of the time. $\mathbb{P}(A|\bar{B})$

If you pick up a bar and it alerts, what is the probability you have a golden ticket? $\mathbb{P}(B|A)$

Reversing the Conditioning

All of our information conditions on whether B happens or not – does your bar have a golden ticket or not?

But we're interested in the "reverse" conditioning. We know the scale alerted us – we know the test is positive – but do we have a golden ticket?

Bayes' Rule

Bayes' Rule

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

Bayes' Rule

Bayes' Rule

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

What do we know about Wonka Bars?

$$0.999 = \frac{\mathbb{P}(B|A) \cdot \mathbb{P}(A)}{.001}$$

Filling In

What's $\mathbb{P}(A)$?

We'll use a trick called "the law of total probability":

$$\mathbb{P}(A) = \mathbb{P}(A|B) \cdot \mathbb{P}(B) + \mathbb{P}(A|\bar{B}) \cdot P(\bar{B})$$

$$= 0.999 \cdot .001 + .01 \cdot .999$$

$$= .010989$$

Bayes' Rule

What do we know about Wonka Bars?

$$0.999 = \frac{\mathbb{P}(B|A) \cdot .010989}{.001}$$

Solving $\mathbb{P}(B|A) = \frac{1}{11}$, i.e. about 0.0909.

Only about a 10% chance that the bar has the golden ticket!