Continuous Random Variables Continued

CSE 312 Summer 21
Lecture 15

Announcements

Only 1 person signed up for extra OH! 😢

Extending the sign up for OH.

Please sign up by tonight.

Continuous Zoo

$X \sim \text{Unif}(a, b)$	$X \sim \text{Exp}(\lambda)$	$X \sim \mathcal{N}(\mu, \sigma^2)$
$f_X(k) = \frac{1}{b-a}$ $\mathbb{E}[X] = \frac{a+b}{2}$ $\text{Var}(X) = \frac{(b-a)^2}{12}$	$f_X(k) = \lambda e^{-\lambda k}$ for $k \geq 0$ $\mathbb{E}[X] = \frac{1}{\lambda}$ $\text{Var}(X) = \frac{1}{\lambda^2}$	$f_X(k) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ $\mathbb{E}[X] = \mu$ $\text{Var}(X) = \sigma^2$

Uniform Distribution (Continuous)

Continuous Uniform Distribution

$X \sim \text{Unif}(a, b)$ (uniform real number between a and b)

PDF: $f_X(k) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq k \leq b \\ 0 & \text{otherwise} \end{cases}$

CDF: $F_X(k) = \begin{cases} 0 & \text{if } k < a \\ \frac{k-a}{b-a} & \text{if } a \leq k \leq b \\ 1 & \text{if } k \geq b \end{cases}$

$\mathbb{E}[X] = \frac{a+b}{2}$

$\text{Var}(X) = \frac{(b-a)^2}{12}$

Exponential Distribution

Exponential Random Variable

Like a geometric random variable, but continuous time. How long do we wait until an event happens? (instead of "how many flips until a heads")

Where waiting doesn't make the event happen any sooner.

Geometric: $\mathbb{P}(X = k + 1 \,|X \geq 1) = \mathbb{P}(X = k)$

When the first flip is tails, the coin doesn't remember it came up tails, you've made no progress.

For an exponential random variable:

$\mathbb{P}(Y \geq k + 1 \,|Y \geq 1) = \mathbb{P}(Y \geq k)$

Exponential Random Variable

If you take a Poisson random variable and ask "what's the time until the next event" you get an exponential distribution!

Let's find the CDF for an exponential.

Let $Y \sim \text{Exp}(\lambda)$, be the time until the first event, when we see an average of λ events per time unit. What's $\mathbb{P}(Y > t)$?

What Poisson are we waiting on? For $X \sim \text{Poi}(\lambda t)$ $\mathbb{P}(Y > t) = \mathbb{P}(X = 0)$

$\mathbb{P}(X = 0) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$

$F_Y(t) = \mathbb{P}(Y \le t) = 1 - e^{-\lambda t}$ (for $t \ge 0$, $F_Y(x) = 0$ for $x < 0$)

Find the density

We know the CDF, $F_Y(t) = \mathbb{P}(Y \leq t) = 1 - e^{-\lambda t}$

What's the density?

$f_Y(t) =$

Find the density

We know the CDF, $F_Y(t) = \mathbb{P}(Y \leq t) = 1 - e^{-\lambda t}$

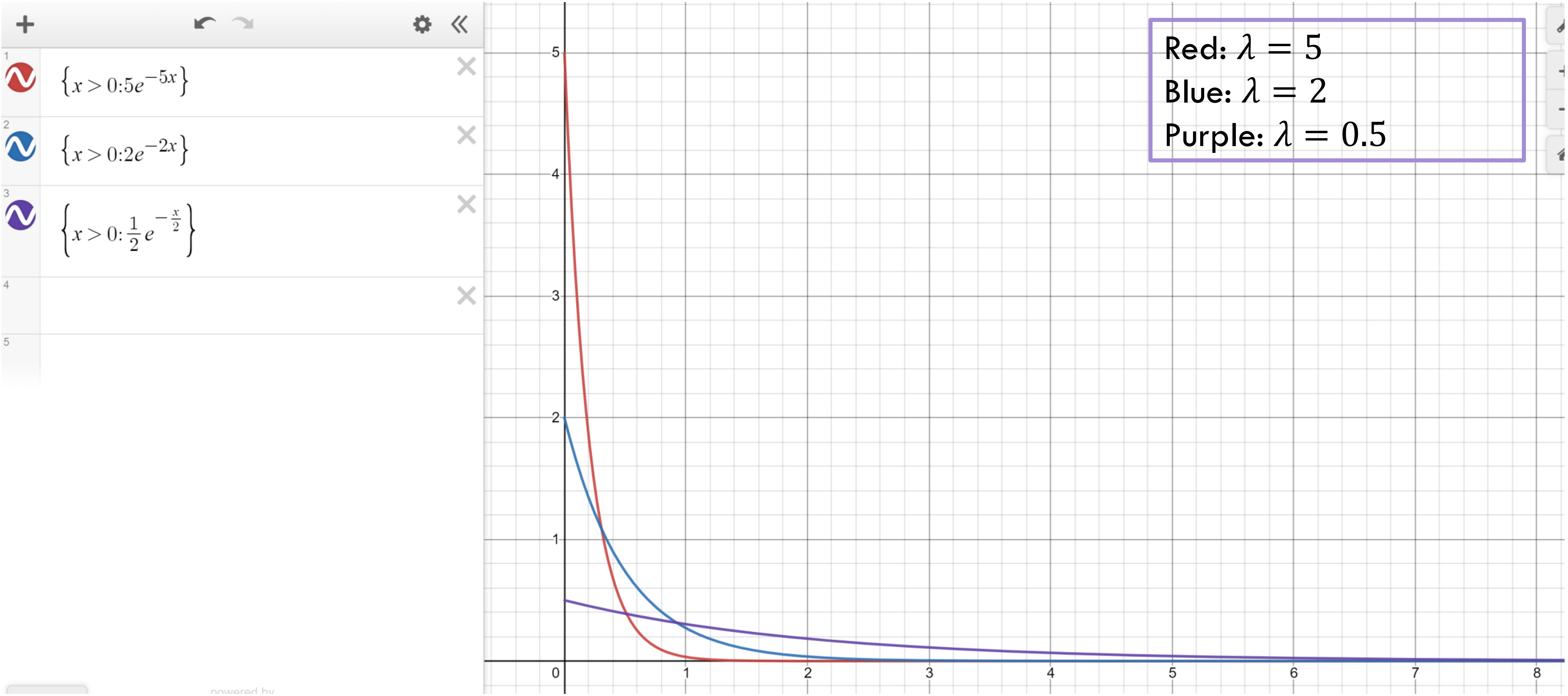
What's the density?

$$f_Y(t) = \frac{d}{dt}\left(1 - e^{-\lambda t}\right) = 0 - \frac{d}{dt}\left(e^{-\lambda t}\right) = \lambda e^{-\lambda t}.$$

For t ≥ 0 it's that expression

For $t < 0$ it's just $\mathbf{0}$.

Exponential PDF



Memorylessness

$$\mathbb{P}(X \geq k+1 | X \geq 1) = \frac{\mathbb{P}(X \geq k+1 \cap X \geq 1)}{\mathbb{P}(X \geq 1)} = \frac{\mathbb{P}(X \geq k+1)}{1-(1-e^{-\lambda \cdot 1})}$$

$$= \frac{e^{-\lambda(k+1)}}{e^{-\lambda}} = e^{-\lambda k}$$

What about $\mathbb{P}(X \geq k)$ (without conditioning on the first step)?

$1-(1-e^{-\lambda k}) = e^{-\lambda k}$

It’s the same!!!

More generally, for an exponential rv X, $\mathbb{P}(X \geq s+t | X \geq s) = \mathbb{P}(X \geq t)$

Side note

I hid a trick in that algebra,

$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X < 1) = 1 - \mathbb{P}(X \leq 1)$

The first step is the complementary law.

The second step is using that $\int_1^1 f_X(z)\mathrm{d}z = 0$

In general, for continuous random variables we can switch out $\leq$ and $<$ without anything changing.

We can't make those switches for discrete random variables.

Expectation of an exponential

Don't worry about the derivation (it's here if you're interested; you're not responsible for the derivation. Just the value.

Let $X \sim \text{Exp}(\lambda)$

$\mathbb{E}[X] = \int_{-\infty}^{\infty} z \cdot f_X(z)\text{dz}$

$= \int_0^{\infty} z \cdot \lambda e^{-\lambda z}\, dz$

Let $u = z;\ dv = \lambda e^{-\lambda z} dz\ (v = -e^{-\lambda z})$

Integrate by parts: $-ze^{-\lambda z} - \int -e^{-\lambda z}\, dz = -ze^{-\lambda z} - \frac{1}{\lambda} e^{-\lambda z}$

Definite Integral: $-ze^{-\lambda z} - \frac{1}{\lambda} e^{-\lambda z}|_{z=0}^{\infty} = (\lim_{z\to\infty} -ze^{-\lambda z} - \frac{1}{\lambda} e^{-\lambda z})\ - (0 - \frac{1}{\lambda})$

By L'Hopital's Rule $(\lim_{z\to\infty} -\frac{z}{e^{\lambda z}} - \frac{1}{\lambda e^{\lambda z}})\ - (0 - \frac{1}{\lambda}) = \left(\lim_{z\to\infty} -\frac{1}{\lambda e^{\lambda z}}\right) + \frac{1}{\lambda} = \frac{1}{\lambda}$

Variance of an exponential

If $\mathrm{X} \sim \mathrm{Exp}(\lambda)$ then $\mathrm{Var}(X) = \frac{1}{\lambda^2}$

Similar calculus tricks will get you there.

Exponential

$X \sim \mathrm{Exp}(\lambda)$

Parameter $\lambda \geq 0$ is the average number of events in a unit of time.

$$f_X(k) = \begin{cases} \lambda e^{-\lambda k} & \text{if } k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(k) = \begin{cases} 1 - e^{-\lambda k} & \text{if } k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}$$

Normal Distribution

Normal Random Variable

X is a normal (aka Gaussian) random variable with mean μ and variance σ^2 (written $X \sim \mathcal{N}(\mu, \sigma^2)$) if it has the density:

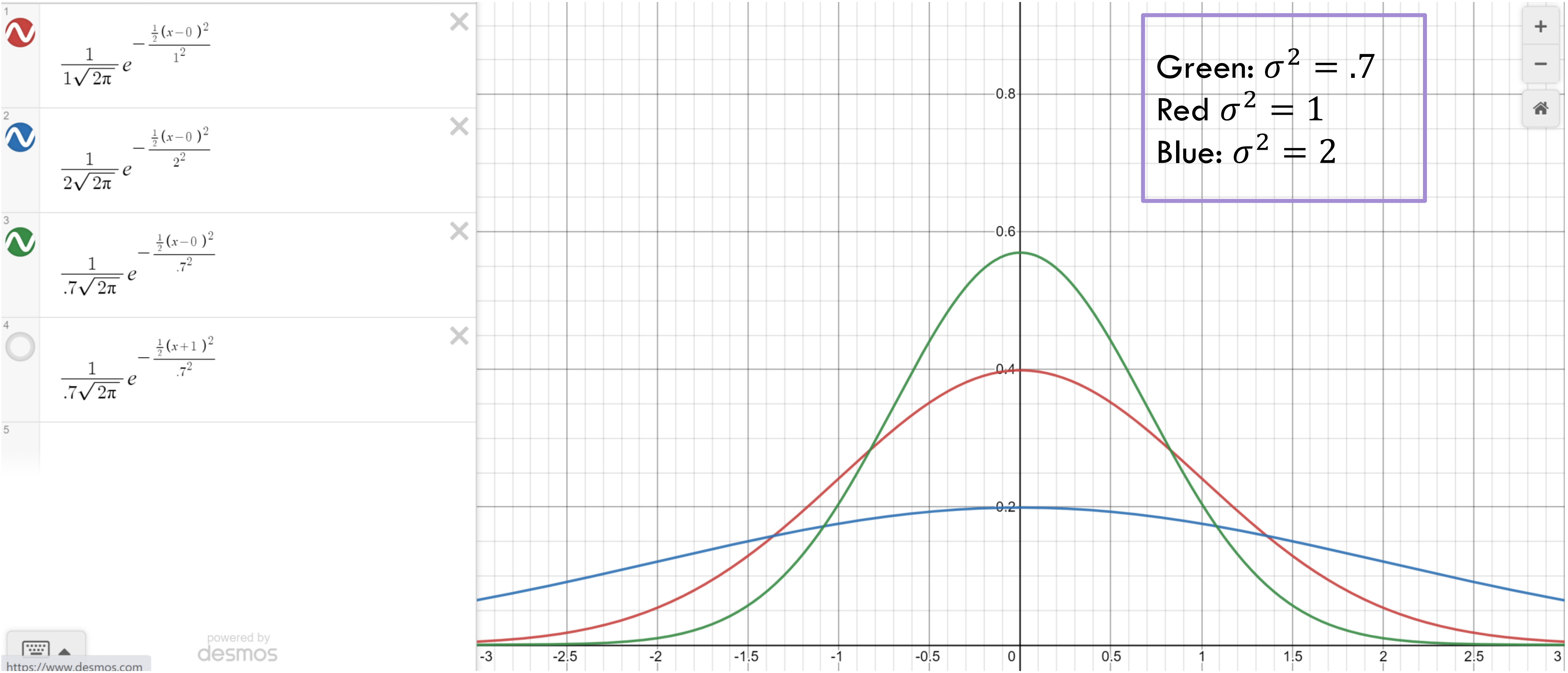
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Let's get some intuition for that density…

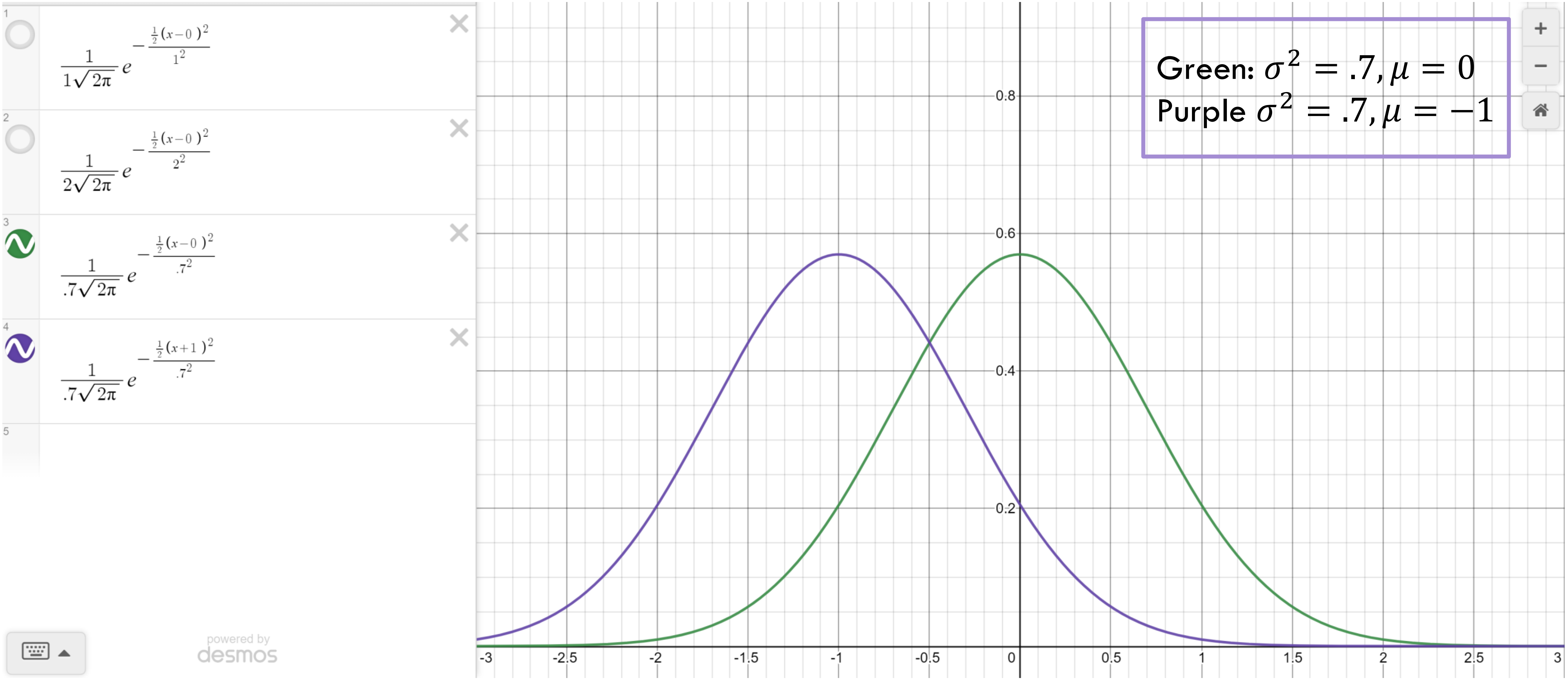
Is $\mathbb{E}[X] = \mu$?

Yes! Plug in $\mu - k$ and $\mu + k$ and you'll get the same density for every k. The density is symmetric around μ. The expectation must be μ.

Changing the variance



Changing the mean



Scaling Normals

When we scale a normal (multiplying by a constant or adding a constant) we get a normal random variable back!

If $X \sim \mathcal{N}(\mu, \sigma^2)$

Then for $Y = aX + b$, $Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Normals are unique in that you get a NORMAL back.

If you multiply a binomial by $3/2$ you don't get a binomial (it's support isn't even integers!)

Normalize

To turn $\mathrm{X} \sim \mathcal{N}(\mu, \sigma^2)$ into $\mathrm{Y} \sim \mathcal{N}(0,1)$ you want to set

$Y = \frac{X-\mu}{\sigma}$

Why normalize?

The density is a mess. The CDF does not have a pretty closed form.

But we're going to need the CDF a lot, so…

Table of Standard Normal CDF

The way we'll evaluate the CDF of a normal is to:
1. convert to a standard normal
2. Round the "z-score" to the hundredths place.
3. Look up the value in the table.

It's 2021, we're using a table?

The table makes sure we have consistent rounding rules (makes it easier for us to debug with you).
You can't evaluate this by hand – the "z-score" can give you intuition right away.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Use the table!

We'll use the notation $\Phi(z)$ to mean $F_X(z)$ where $X \sim \mathcal{N}(0,1)$.

Let $Y \sim \mathcal{N}(5,4)$ what is $\mathbb{P}(Y > 9)$?

$\mathbb{P}(Y > 9)$

$= \mathbb{P}\left(\frac{Y-5}{2} > \frac{9-5}{2}\right)$ we've just written the inequality in a weird way.

$= \mathbb{P}(X > \frac{9-5}{2})$ where X is $\mathcal{N}(0,1)$.

$= 1 - \mathbb{P}\left(X \leq \frac{9-5}{2}\right) = 1 - \Phi(2.00) = 1 - 0.97725 = 0.02275.$

More Practice

Let $X \sim \mathcal{N}(3, 2)$

What is the probability of $1 \leq X \leq 4$?

More Practice

Let $X \sim \mathcal{N}(3, 2)$

What is the probability of $1 \le X \le 4$?

$\mathbb{P}(1 \le X \le 4)$

$= \mathbb{P}\left(\frac{1-3}{\sqrt{2}} \le \frac{X-3}{\sqrt{2}} \le \frac{4-3}{\sqrt{2}}\right)$

$= \mathbb{P}\left(-1.41 \le \frac{X-3}{\sqrt{2}} \le 0.71\right)$

$= \Phi(0.71) - \Phi(-1.41)$

$= \Phi(0.71) - \left(1 - \Phi(1.41)\right) = 0.76115 - (1 - 0.92073) = 0.68188$

In real life

What’s the probability of being at most two standard deviations from the mean?

In real life

What's the probability of being at most two standard deviations from the mean?

$$= \Phi(2) - \Phi(-2)$$

$$= \Phi(2) - \left(1 - \Phi(2)\right)$$

$$= 0.97725 - (1 - 0.97725) = 0.9545$$

You may hear this being referred to as the "68-95-99.7 rule" which is the probability of being within 1, 2, and 3 standard deviations of the mean.