Quiz Section 1 – Solutions

Review

1) Sum rule. If you can choose from EITHER one of n options, OR one of m options with NO overlap with the previous n, then the number of possible outcomes of the experiment is ____________________.

2) Product rule. In a sequential process with m steps, if there are n_1 choices for the 1st step, n_2 choices for the 2nd step (given the first choice), ..., and n_m choices for the mth step (given the previous choices), then the total number of outcomes is ____________________.

3) Permutations. The number of ways to re-order n elements is ________.

4) k-permutations. The number of ways to choose a sequence of k distinct elements from a set of n elements is _____.

5) Set difference. Is it always true that $|A \backslash B| = |A| - |B|$?

Task 1 – Sets

a) For each one of the following sets, give its **cardinality**, i.e., indicate how many elements it contains:

- $A = \varnothing$
- $B = \{\varnothing\}$
- $C = \{\{\varnothing\}\}$
- $D = \{\varnothing, \{\varnothing\}\}$

- $|A| = 0$
- $|B| = 1$
- $|C| = 1$
- $|D| = 2$

b) Let $S = \{a, b, c\}$ and $T = \{c, d\}$. Compute:

- $S \cup T$
- $S \cap T$
- $S \backslash T$
- $2^{S \backslash T}$
- $S \times T$

- $S \cup T = \{a, b, c, d\}$
- $S \cap T = \{c\}$
- $S \backslash T = \{a, b\}$
- $2^{S \backslash T} = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$
- $S \times T = \{(a, c), (a, d), (b, c), (b, d), (c, c), (c, d)\}$

Task 2 – Basic Counting

a) Credit-card numbers are made of 15 decimal digits, and a 16th checksum digit (which is uniquely determined by the first 15 digits). How many credit-card numbers are there?

10^{15}

b) How many positive divisors does $1440 = 2^5 3^2 5$ have?

$6 \cdot 3 \cdot 2 = 36.$

Every positive divisor of 1440 can be written as $2^i 3^j 5^k$ where $i \in \{0, \ldots, 5\}$, $j \in \{0, 1, 2\}$, and $k \in \{0, 1\}$.

c) How many ways are there to arrange the CSE 312 staff on a line (12 TAs, one professor) for a group picture?

$13!.$

d) How many ways are there to arrange the CSE 312 staff on a line so that Professor Beame is at one of the ends or exactly in the middle?

$3 \times 12!.$

Three options for the choice for Professor Beame and $12!$ ways to order the TAs.

Task 3 – Seating

How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if ...

a) ... all couples are to get adjacent seats?

Consider each couple as a unit. Apply the product rule, first choosing one of the $5!$ permutations of the 5 couples, and then, for each couple in turn, choosing one of the 2 permutations for how they sit (for a total of 2^5). Therefore, the answer is: $\boxed{5! \cdot 2^5.}$

b) ... anyone can sit anywhere, except that one couple insists on *not* sitting in adjacent seats?

Apply complementary counting to first compute the total number of arrangements of the 10 people, and then subtract from this the number of arrangements in which that particular couple does get adjacent seats. There are $10!$ for the former, and there are $9! \cdot 2$ arrangements in which this couple does sit in adjacent seats, since you can treat the couple as a unit, permute the 9 "individuals" (consisting of 8 people plus the couple) and then consider the 2 permutations for that couple. That means the answer to the question is $\boxed{10! - 9! \cdot 2 = 8 \cdot 9!}$.

Alternatively, we can do casework. Name the two people in the couple A and B. There are two cases: A can sit on one of the ends, or not. If A sits on an end seat, A has 2 choices and B has 8 possible seats. If A doesn't sit on the end, A has 8 choices and B only has 7. So there are a total of $2 \cdot 8 + 8 \cdot 7$ ways A and B can sit. Once they do, the other 8 people can sit in $8!$ ways since there are no other restrictions. Hence the total number of ways is $\boxed{(2 \cdot 8 + 8 \cdot 7)8! = 9 \cdot 8 \cdot 8! = 8 \cdot 9!}$.

Task 4 – Weird Card Game

In how many ways can a pack of fifty-two cards (in four suits of thirteen cards each) be dealt to thirteen players, four to each, so that every player has one card from each of the suits?

> Apply the product rule: First deal the hearts, one to each person, then the spades, one to each person, then diamonds, then the clubs. For each of these steps, there are $13!$ possibilities. Therefore, the answer is $\boxed{13!^4}$.

Task 5 – HBCDEFGA

How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

> The total number of permutations is $8!$. The number of permutations with A at the beginning is $7!$ and the number with H at the end is $7!$. By inclusion/exclusion, the number that have either A at the beginning or H at the end or both is $2 \cdot 7! - 6!$ since there are $6!$ that have A at the beginning and H at the end. Finally, using complementary counting, the number that have neither A at the end or H at the end is $8! - (2 \cdot 7! - 6!)$.

Task 6 – Lizards and Snakes!

Loudon has three pet lizards, Rango, a gecko named Gordon, and a goanna named Joanna, as well as two small pet snakes, Kaa and Basilisk, but only 4 terrariums to put them in. In how many different ways can he put his 5 pets in these 4 terrariums so that no terrarium has both a snake and a lizard?

> Find the number of possibilities when Kaa and Basilisk go in the same terrarium, and find the number of possibilities when they go to different terrariums, and then use the sum rule to get the final answer.
>
> If Kaa and Basilisk go in the same terrarium, there are 4 terrarium it could be. For each such choice, there are 3 choices of terrarium for each of the 3 lizards, so 3^3 choices for all the lizards.
>
> If Kaa and Basilisk go in different terrariums, there are $4 \cdot 3 = 12$ pairs of terrariums they could go in. For each such choice, there are 2 choices of terrarium for each of the 3 lizards, so 2^3 choices for all the lizards.
>
> Therefore the answer is $\boxed{4 \cdot 3^3 + 12 \cdot 2^3}$.

Task 7 – Birthday Cake

A chef is preparing desserts for the week, starting on a Sunday. On each day, only one of five desserts (apple pie, cherry pie, strawberry pie, pineapple pie, and cake) may be served. On Thursday there is a birthday, so cake must be served that day. On no two consecutive days can the chef serve the same dessert. How many dessert menus are there for the week?

Apply the product rule. Start from Thursday and work forward and backward in the week:

More precisely, given the 1 choice on Thursday, for each of Wednesday and Friday, there are 4 choices (the different pie options). Given the choice on Wednesday, there are 4 choices for Tuesday, and given the choice on Tuesday, there are 4 choices for Monday, and given the choice on Monday, there are 4 choices on Sunday. Similarly, given the choice on Friday, there are 4 choices on Saturday.

Therefore the answer is $4 \cdot 4 \cdot 4 \cdot 4 \cdot 1 \cdot 4 \cdot 4 = \boxed{4^6}$

Task 8 – Photographs

Suppose that 8 people, including you and a friend, line up for a picture. In how many ways can the photographer organize the line if she wants to have fewer than 2 people between you and your friend?

This is most easily solved using the sum rule. Count the number of ways the line can be organized if you are next to your friend. Then count the number of ways the line can be organized if there is one person between you and your friend. Then use the sum rule to add these up.

Case 1: You are next to your friend. So we can think of you and your friend as being a "unit". Now apply the product rule: there are $7!$ ways to arrange the other 6 people together with the unit (of you and your friend). Once arranged, there are 2 ways to rearrange you and your friend in the order. So there are $7! \cdot 2$ ways to to line people up if you are next to your friend.

Case 2: There is exactly 1 person between you and your friend. Apply the product rule by first picking the person who is between you (6 choices). Then, thinking of you, your friend and that person as a "unit", consider all arrangements of the 5 people plus the unit ($6!$ ways). Finally, there are two ways for you and your friend to be placed within the trio. Therefore, altogether there are $6 \cdot 6! \cdot 2$ possibilities.

Therefore, the final answer is $\boxed{(2 \cdot 7 + 2 \cdot 6) \cdot 6!}$

Task 9 – Extended Family Portrait

A group of n families, each with m members, are to be lined up for a photograph. In how many ways can the nm people be arranged if members of a family must stay together?

Apply the product rule. First order the families; there are $n!$ ways to do this. Then consider the families one by one and reorder their members. Within each family, there are $m!$ ways to order their members. So there are a total of $\boxed{n!(m!)^n}$ ways to line these people up according to the given constraints.