CSE 312

Foundations of Computing II

Lecture 3: Even more counting

Binomial Theorem, Inclusion-Exclusion, Pigeonhole Principle

Recap

Two core rules for counting a set S:

- Sum rule:
  - Break up S into disjoint pieces/cases
  - $|S|$ = the sum of the sizes of the pieces.

- Product rule:
  - View the elements of S as being constructed by a series of choices, where the # of possibilities for each choice doesn't depend on the previous choices
  - $|S|$ = the product of the # of choices in each step of the series.

Recap

- k-sequences: How many length k sequences over alphabet of size n?
  - Product rule ➔ n^k
- k-permutations: How many length k sequences over alphabet of size n, without repetition?
  - Permutation ➔ $\frac{n!}{(n-k)!}$
- k-combinations: How many size k subsets of a set of size n (without repetition and without order)?
  - Combination ➔ $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Binomial Coefficients – Many interesting and useful properties

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad \binom{n}{n} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{0} = 1$$

Fact. $\binom{n}{k} = \binom{n}{n-k}$ — **Symmetry in Binomial Coefficients**

Fact. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ — **Follows from Binomial Theorem**

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ — **Pascal's Identity**

Binomial Theorem: Idea

Poll: What is the coefficient for xy^3?

A. 4 ✓

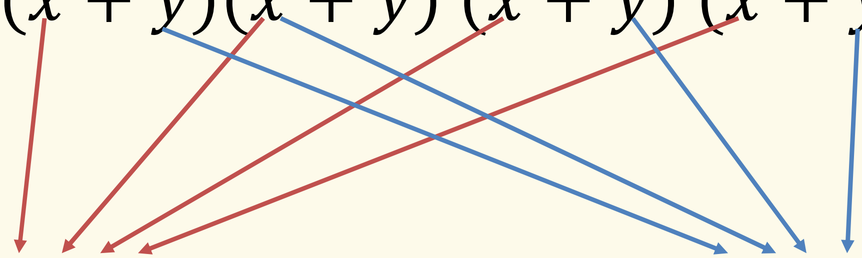
B. $\binom{4}{1}$ ✓

C. $\binom{4}{3}$ ✓

D. 3

$$(x+y)^2 = (x+y)(x+y)$$
$$= xx + xy + yx + yy$$
$$= x^2 + 2xy + y^2$$

$$(x+y)^4 = (x+y)(x+y)\,(x+y)\,(x+y)$$



$$= xxxx + yyyy + xyxy + yxyy + \ \ldots$$

Binomial Theorem: Idea

$$(x + y)^n = (x + y) \dots (x + y)$$

Each term is of the form $x^k y^{n-k}$, since each term is made by multiplying exactly n variables, either x or y, one from each copy of $(x + y)$

How many times do we get $x^k y^{n-k}$?

The number of ways to choose x from exactly k of the n copies of $(x + y)$ (the other $n - k$ choices will be y) which is:

$$\binom{n}{k} = \binom{n}{n-k}$$

Binomial Theorem

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Apply with $x = y = 1$

Many properties of sums of binomial coefficients can be found by plugging in different values of x and y in the Binomial Theorem.

Corollary.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Agenda

- Binomial Theorem
- Combinatorial Proofs
- Inclusion-Exclusion
- Pigeonhole Principle
- Counting Practice

Recall: Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

1. Choose which k elements are included
2. Choose which $n - k$ elements are excluded

Format for a combinatorial argument/proof of $a = b$

- Let S be a set of objects
- Show how to count $|S|$ one way $\Rightarrow$ $|S| = a$
- Show how to count $|S|$ another way $\Rightarrow$ $|S| = b$

Combinatorial argument/proof

- Elegant
- Simple
- Intuitive

Algebraic argument

- Brute force
- Less Intuitive

Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

How to prove Pascal's identity?

Algebraic argument:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$
$$= 20\ years\ later\ \dots$$
$$= \frac{n!}{k!\,(n-k)!}$$
$$= \binom{n}{k}$$

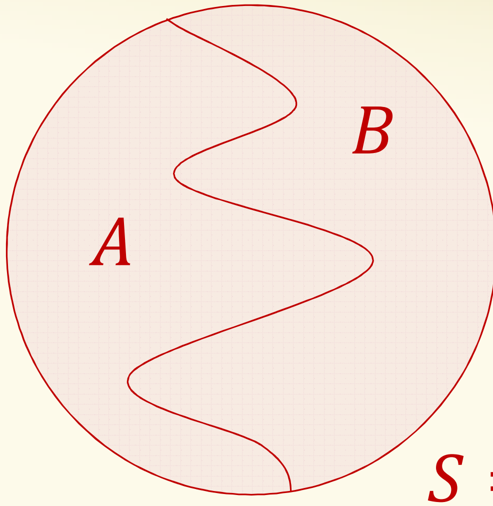
Hard work and not intuitive

Let's see a combinatorial argument

Example – Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$|S| \quad = \quad |A| \quad + \quad |B|$$



$S = A \cup B$

Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have the sizes above.
- The equation then follows by the Sum Rule.

Example – Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$|S| = |A| + |B|$$

Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have these sizes

S: set of size k subsets of $[n] = \{1, 2, \cdots, n\}$.

$|S| = \binom{n}{k}$

e.g. $n = 4, k = 2,$ $S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

A: set of size k subsets of $[n]$ that DO include n

$$A = \{\{1,4\}, \{2,4\}, \{3,4\}\}$$

B: set of size k subsets of $[n]$ that DON'T include n

$$B = \{\{1,2\}, \{1,3\}, \{2,3\}\}$$

Example – Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| \quad = \quad |A| \quad + \quad |B|$

Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have these sizes

S: set of size k subsets of $[n] = \{1, 2, \cdots, n\}$.

n is in set, need to choose other $k-1$ elements from $[n-1]$

$$|A| = \binom{n-1}{k-1}$$

A: set of size k subsets of $[n]$ that DO include n

n not in set, need to choose k elements from $[n-1]$

$$|B| = \binom{n-1}{k}$$

B: set of size k subsets of $[n]$ that DON'T include n

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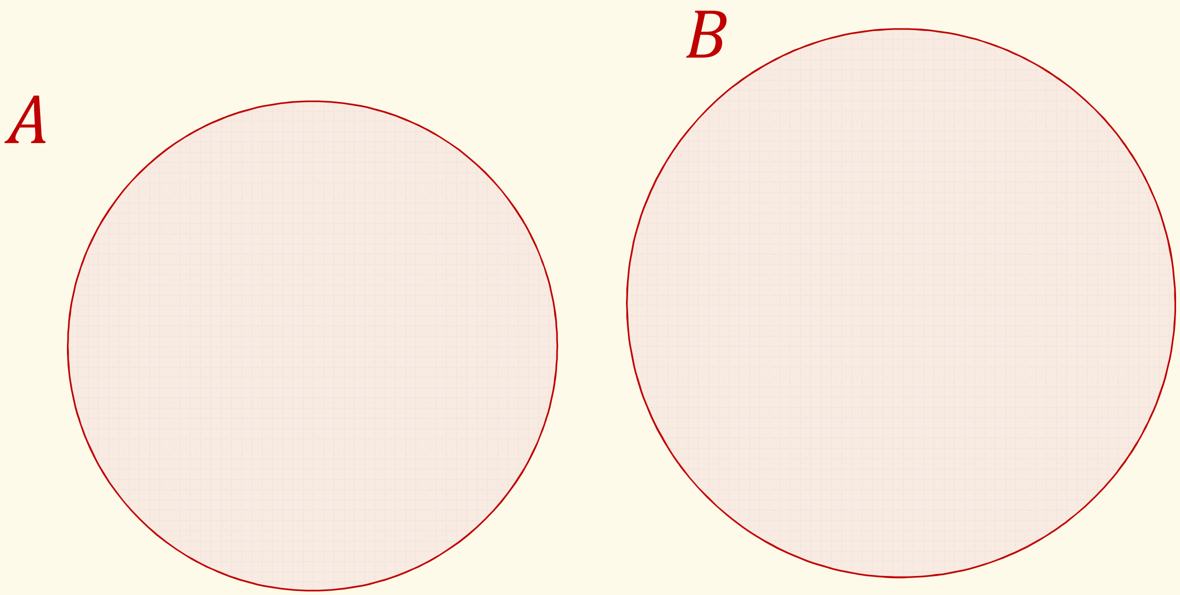
Recap Disjoint Sets

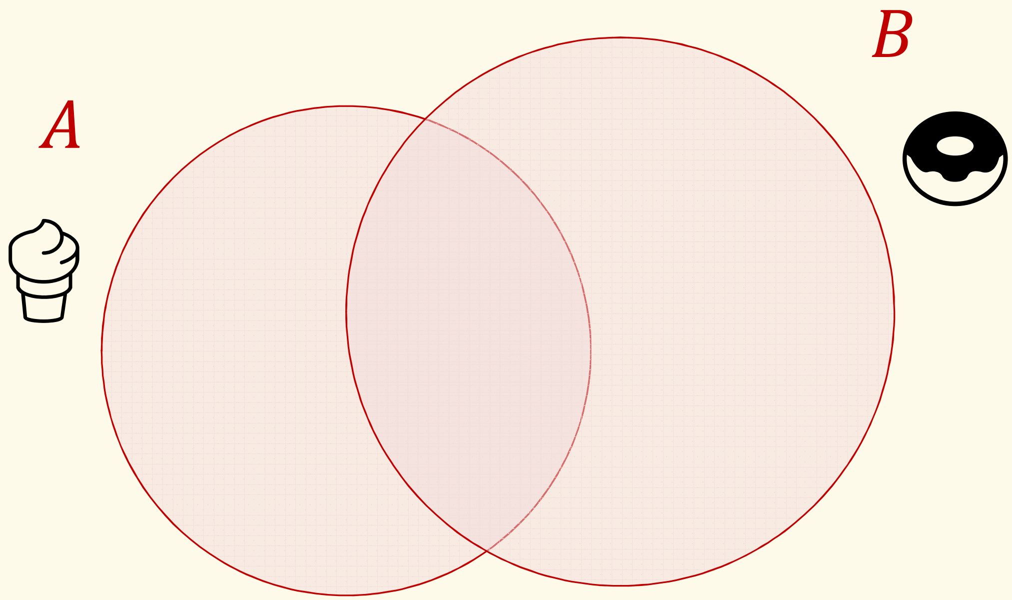
Sets that do not contain common elements ($A \cap B = \emptyset$)



Sum Rule: $|A \cup B| = |A| + |B|$

Inclusion-Exclusion

But what if the sets are not disjoint?



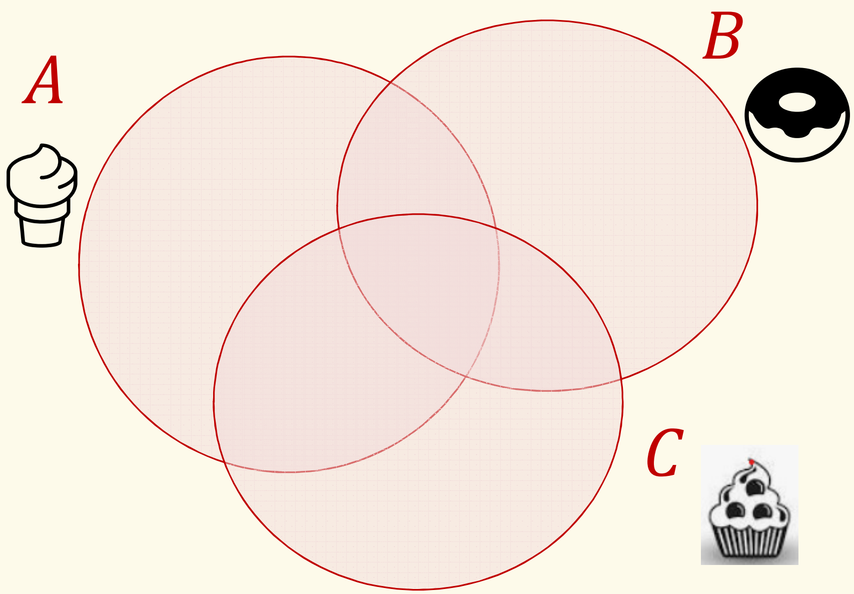
$|A| = 43$
$|B| = 20$
$|A \cap B| = 7$
$|A \cup B| = ???$

Fact. $|A \cup B| = |A| + |B| - |A \cap B|$

Inclusion-Exclusion

Not drawn to scale

What if there are three sets?



$|A| = 43$
$|B| = 20$
$|C| = 35$
$|A \cap B| = 7$
$|A \cap C| = 16$
$|B \cap C| = 11$
$|A \cap B \cap C| = 4$
$|A \cup B \cup C| = ???$

Fact.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Inclusion-Exclusion

Let A, B be sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

In general, if $A_1, A_2, \dots, A_n$ are sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = singles - doubles + triples - quads + \ \dots$$
$$= (|A_1| + \cdots + |A_n|) - (|A_1 \cap A_2| + \ \dots + |A_{n-1} \cap A_n|) + \ \dots$$

Brain Break

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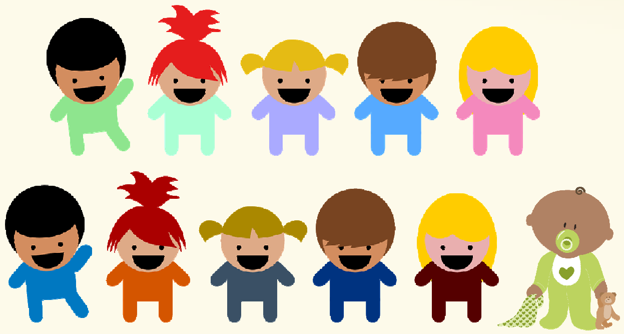
Pigeonhole Principle (PHP): Idea

10 pigeons, 9 pigeonholes

Pigeonhole Principle: Idea



If 11 children have to share 3 cakes, at least one cake must be shared by how many children?

Pigeonhole Principle – More generally

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\frac{n}{k}$ pigeons!

Proof. Assume there are $< \frac{n}{k}$ pigeons per hole.

Then, there are $< k \cdot \frac{n}{k} = n$ pigeons overall.

Contradiction!

Pigeonhole Principle – Better version

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons!

Reason. Can't have fractional number of pigeons

Syntax reminder:

- Ceiling: $\lceil x \rceil$ is x rounded up to the nearest integer (e.g., $\lceil 2.731 \rceil = 3$)
- Floor: $\lfloor x \rfloor$ is x rounded down to the nearest integer (e.g., $\lfloor 2.731 \rfloor = 2$)

Pigeonhole Principle: Strategy

To use the PHP to solve a problem, there are generally 4 steps

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

Pigeonhole Principle – Example

In a room with 367 people, there are at least two with the same birthday.

Solution:

1. **367** pigeons = people
2. **366** holes (365 for a normal year + Feb 29) = possible birthdays
3. Person goes into hole corresponding to own birthday
4. By PHP, there must be two people with the same birthday

Pigeonhole Principle – Example (Surprising?)

*In every set S of 100 integers, there are at least **two** elements whose difference is a multiple of 37.*

When solving a PHP problem:

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

Pigeonhole Principle – Example (Surprising?)

*In every set S of 100 integers, there are at least **two** elements whose difference is a multiple of 37.*

When solving a PHP problem:

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

Pigeons: integers x in S

Pigeonholes: $\{0,1,\ldots,36\}$

Assignment: x goes to $x \bmod 37$

Since $100 > 37$, by PHP, there are $x \neq y \in S$ s.t.

$x \bmod 37 = y \bmod 37$ which implies

$x - y = 37\,k$ for some integer k

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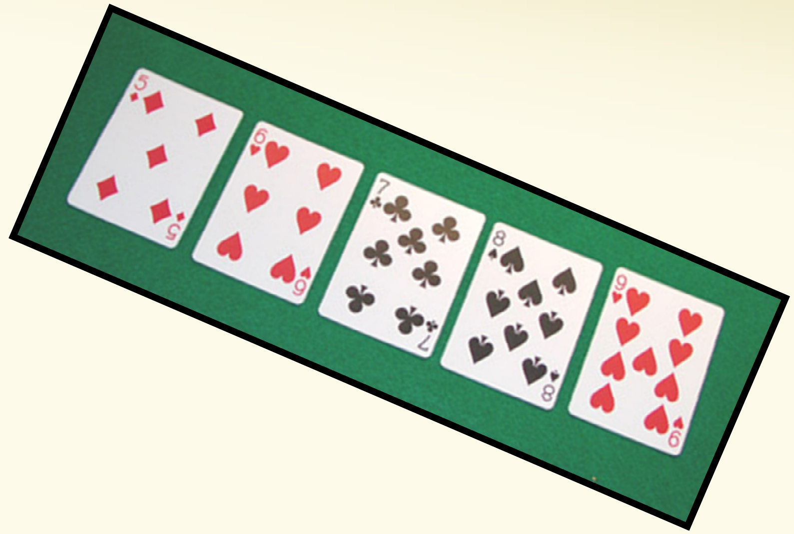
Quick Review of Cards



How many possible 5 card hands?

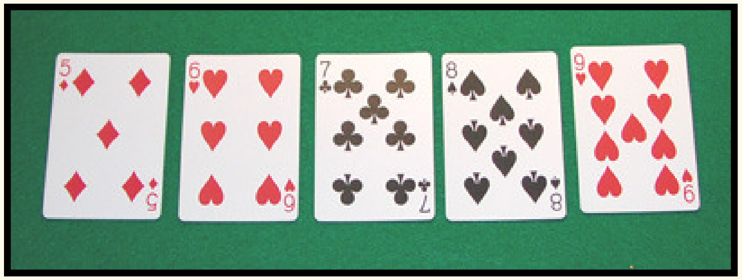
$$\binom{52}{5}$$

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

Counting Cards I

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

- A "straight" is five consecutive rank cards of any suit (where A,2,3,4,5 also counts as consecutive).
  How many possible straights?



$$10 \cdot 4^5 = 10{,}240$$

Counting Cards II

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

- A flush is five card hand all of the same suit.

How many possible flushes?

$$4 \cdot \binom{13}{5} = 5148$$

Counting Cards III

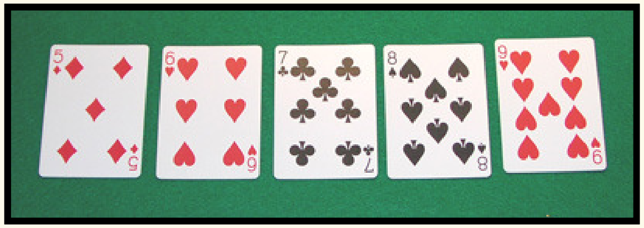
- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

- A flush is five card hand all of the same suit.

How many possible flushes?

$$4 \cdot \binom{13}{5} = 5148$$

- How many flushes are NOT straights?



= #flush - #flush and straight

$$\left(4 \cdot \binom{13}{5} = 5148\right) - 10 \cdot 4$$

Sleuth's Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence → under counting   Many sequences → over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards.

$$\binom{4}{3} \cdot \binom{49}{2}$$

<u>Poll:</u>

A. Correct
B. Overcount
C. Undercount

https://pollev.com/paulbeame028

Sleuth's Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

Many sequences → over counting

EXAMPLE: How many ways are there to choos[e] ... contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards.

$$\binom{4}{3} \cdot \binom{49}{2}$$

Problem: This counts a hand with all 4 Aces in 4 different ways!
*e.g. it counts **A♣, A♦, A♥, A♠, 2♥*** ***four** times:*

{A♣, A♦, A♥} {A♠, 2♥}
{A♣, A♦, A♠} {A♥, 2♥}
{A♣, A♥, A♠} {A♦, 2♥}
{A♦, A♥, A♠} {A♣, 2♥}

Sleuth's Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence ➔ under counting   Many sequences ➔ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

Use the sum rule

= # 5 card hand containing exactly 3 Aces $\binom{4}{3} \cdot \binom{48}{2}$

+ # 5 card hand containing exactly 4 Aces $\binom{48}{1}$

Counting when order only *partly* matters

We often want to count # of partly ordered lists:

Let M = # of ways to produce fully ordered lists

P = # of partly ordered lists

N = # of ways to produce corresponding fully ordered list given a partly ordered list

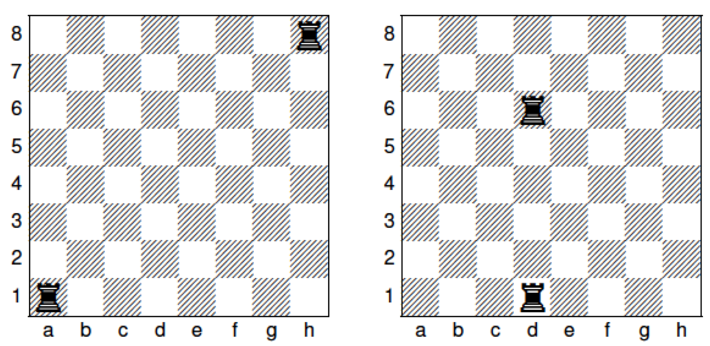
Then $M = P \cdot N$ by the product rule. Often M and N are easy to compute:

$$P = M/N$$

Dividing by N "removes" part of the order.

Rooks on chessboard

How many ways to place two identical rooks on a chessboard so that they don't share a row or a column



(a) valid (b) invalid

Fully ordered: Pretend Rooks are different

1. Column for rook1
2. Row for rook1
3. Column for rook2
4. Row for rook2

$$(8 \cdot 7)^2$$

"Remove" the order of the two rooks:

$$(8 \cdot 7)^2/2$$

Binomial Theorem

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Set $x = y = 1$

Corollary.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Binomial Theorem: A less obvious consequence

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$= -1$ if k is odd
$= +1$ if k is even

Corollary. For every n, if O and E are the sets of odd and even integers between 0 and n

$$\sum_{k \in O} \binom{n}{k} = \sum_{k \in E} \binom{n}{k}$$

e.g., n=4: 1 4 6 4 1

Proof: Set $x = -1, y = 1$ in the binomial theorem

Tools and concepts

- Sum rule, Product rule
- Permutations, combinations
- Inclusion-exclusion
- Binomial Theorem
- Combinatorial proofs
- Pigeonhole principle
- Binary encoding/stars and bars