CSE 312

Foundations of Computing II

Lecture 4: Intro to Discrete Probability

Announcement

- PSet 1 due tonight
- PSet 2 posted this evening, due next Wednesday

Before probability a quick wrap-up for counting…

Final count: A less obvious consequence of the Binomial Theorem

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$= -1$ if k is odd
$= +1$ if k is even

Corollary. For every n, if O and E are the sets of odd and even integers between 0 and n

$$\sum_{k\in O} \binom{n}{k} = \sum_{k\in E} \binom{n}{k}$$

e.g., n=4: 1 4 6 4 1

Proof: Set $x = -1, y = 1$ in the binomial theorem

Summary of Counting: Tools and concepts

- Sum rule, Product rule
- Permutations, combinations
- Inclusion-exclusion
- Binary encoding/stars and bars
- Pigeonhole principle
- Combinatorial proofs
- Binomial Theorem

Agenda

- Events
- Probability
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- Another Example

Probability

- We want to model a process that is not deterministic.
  - i.e., outcome not determined a-priori
  - E.g. throwing dice, flipping a coin…
  - We want to numerically measure likelihood of outcomes = probability.
  - We want to make complex statements about these likelihoods.
- We will not argue why a certain physical process realizes the probabilistic model we study
  - Why is the outcome of the coin flip really "random"?
- First part of class: "Discrete" probability theory
  - Experiment with finite / discrete set of outcomes.
  - Will explore countably infinite and continuous outcomes later

Sample Space

Definition. A **sample space** Ω is the set of all possible outcomes of an experiment.

Examples:

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Events

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die : $E = \{2, 4, 6\}$

Definition. Events E and F are **mutually exclusive** if $E \cap F = \emptyset$
(i.e., E and F can't happen at same time)

Example:

- For dice rolls: If $E = \{2, 4, 6\}$ and $F = \{1, 5\}$, then $E \cap F = \emptyset$

Example: 4-sided Dice

Suppose I roll blue and red 4-sided dice. Let D1 be the value of the blue die and D2 be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. D1 = 1

B. D1 + D2 = 6

C. D1 = 2 * D2

Die 2 (D2)

Die 1 (D1)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Example: 4-sided Dice

Suppose I roll blue and red 4-sided dice. Let D1 be the value of the blue die and D2 be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. D1 = 1

$A = \{(1,1), (1,2), (1,3), (1,4)\}$

B. D1 + D2 = 6

$B = \{(2,4), (3,3), (4,2)\}$

C. D1 = 2 * D2

$C = \{(2,1), (4,2)\}$

Die 2 (D2) across the top; Die 1 (D1) down the side:

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Example: 4-sided Dice, Mutual Exclusivity

Are A and B mutually exclusive?
How about B and C?
https://pollev.com/stefanotessaro617

	A & B	B & C
(a)	Yes	Yes
(b)	Yes	No
(c)	No	Yes
(d)	No	No

A. D1 = 1

$A = \{(1,1), (1,2), (1,3), (1,4)\}$

B. D1 + D2 = 6

$B = \{(2,4), (3,3), (4,2)\}$

C. D1 = 2 * D2

$C = \{(2,1), (4,2)\}$

Die 2 (D2)

Die 1 (D1)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Agenda

- Events
- **Probability**
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- More Examples

Idea: Probability

A **probability** is a number (between 0 and 1) describing how likely a particular outcome will happen.

Will define a function

$$\mathbb{P} : \Omega \to [0, 1]$$

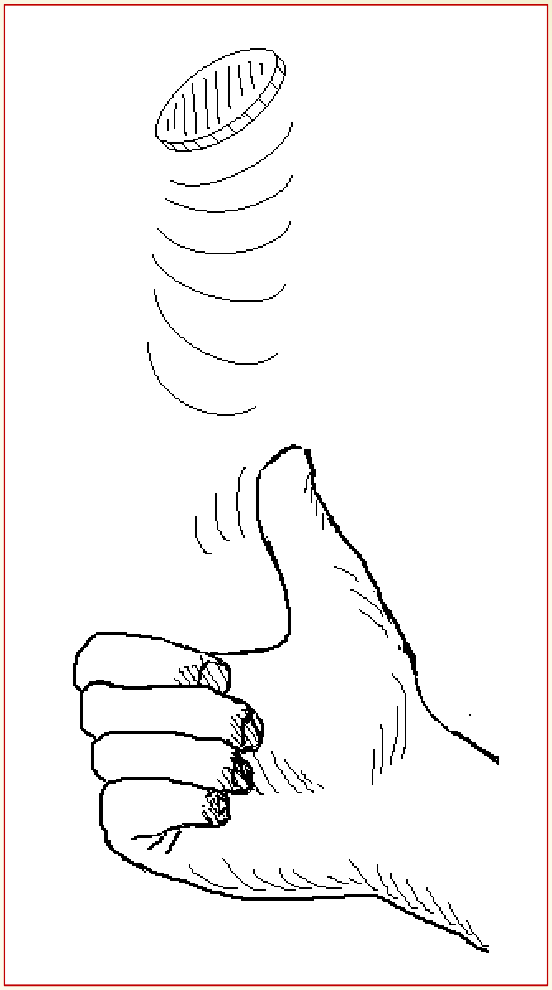
Most written formal CS, math, or stats uses $\mathbb{P}$ or $\Pr$ but for slides we mostly use just P because it is easiest to read

that maps outcomes $\omega \in \Omega$ to probabilities $\mathbb{P}(\omega)$.

– Alternative notations: $\mathbb{P}(\omega) = P(\omega) = \Pr(\omega)$

Example – Coin Tossing

Imagine we toss one coin – outcome can be **heads** or **tails.**



$$\Omega = \{\mathrm{H}, \mathrm{T}\}$$

P? Depends! What do we want to model?!

Fair coin toss

$$P(\mathrm{H}) = P(\mathrm{T}) = \frac{1}{2} = 0.5$$

Example – Coin Tossing

Imagine we toss one coin – outcome can be **heads** or **tails.**

$$\Omega = \{\mathrm{H}, \mathrm{T}\}$$

P? Depends! What do we want to model?!

Bent coin toss (e.g., biased or unfair coin)

$$P(\mathrm{H}) = 0.85, \qquad P(\mathrm{T}) = 0.15$$

Probability space

Either finite or infinite countable (e.g., integers)

Definition. A (discrete) **probability space** is a pair (Ω, P) where:

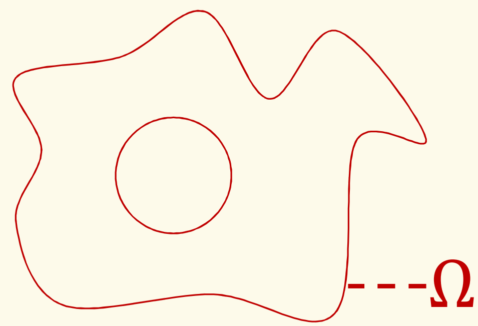
- Ω is a set called the **sample space.**
- P is the **probability measure,**

  a function $P: \Omega \to [0,1]$ such that:

  – $P(x) \geq 0$ for all $x \in \Omega$

  – $\sum_{x \in \Omega} P(x) = 1$

Set of possible **elementary outcomes**



Specify Likelihood (or probability) of each **elementary outcome**

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Uniform Probability Space

Definition. A **<u>uniform</u> probability space** is a pair (Ω, P) such that

$$P(x) = \frac{1}{|\Omega|}$$

for all $x \in \Omega$.

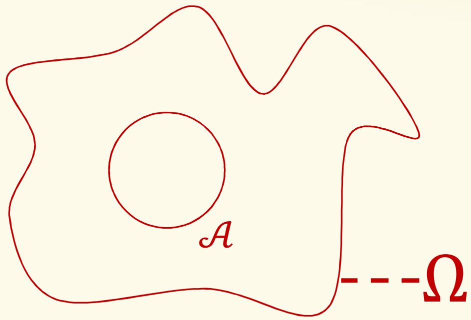
Examples:

- Fair coin $P(x) = \frac{1}{2}$
- Fair 6-sided die $P(x) = \frac{1}{6}$

Events

Definition. An **event** in a probability space (Ω, P) is a subset $\mathcal{A} \subseteq \Omega$. Its probability is

$$P(\mathcal{A}) = \sum_{\omega \in \mathcal{A}} P(\omega)$$



Abuse of notation: When the event $\mathcal{A}$ is a set $\{\omega\}$ with just one outcome ω we write

$P(\omega)$ instead of $P(\{\omega\})$

But that is OK, because they are equal by definition.

Don't care if the argument is an event or outcome!

Agenda

- Events
- Probability
- Equally Likely Outcomes ◀
- Probability Axioms and Beyond Equally Likely Outcomes
- More Examples

Example: 4-sided Dice, Event Probability

Think back to 4-sided die. Suppose each die is fair.

What is the probability of event B? $P(B) = ???$

B. D1 + D2 = 6

$B = \{(2,4), (3,3)(4,2)\}$

$\frac{3}{16}$

Die 2 (D2)

Die 1 (D1)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Equally Likely Outcomes

If (Ω, P) is a **uniform** probability space, then for any event $E \subseteq \Omega$,

$$P(E) = \frac{|E|}{|\Omega|}$$

This follows from the definitions of the probability of an event and uniform probability spaces.

Example – Coin Tossing

Toss a coin 100 times. Each outcome is **equally likely** (and assume the outcome of one toss does not impact another).

What is the probability of seeing 50 heads?

(a) $\frac{1}{2}$

(b) $\frac{1}{2^{50}}$

(c) $\frac{\binom{100}{50}}{2^{100}}$

(d) Not sure

https://pollev.com/paulbeame028

Brain Break

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- More Examples

Axioms of Probability

Let Ω denote the sample space and $E, F \subseteq \Omega$ be events. Note this is applies to ***any*** probability space (not just uniform)

Axiom 1 (Non-negativity): $P(E) \geq 0$.
Axiom 2 (Normalization): $P(\Omega) = 1$.
Axiom 3 (Countable Additivity): If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$

Corollary 1 (Complementation): $P(E^c) = 1 - P(E)$.
Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$.
Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

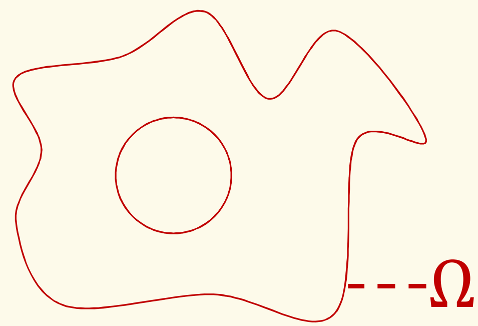
Review Probability space

Either finite or infinite countable (e.g., integers)

Definition. A (discrete) **probability space** is a pair (Ω, P) where:

- Ω is a set called the **sample space.**
- P is the **probability measure,** a function $P: \Omega \to \mathbb{R}$ such that:
  - $P(x) \geq 0$ for all $x \in \Omega$
  - $\sum_{x \in \Omega} P(x) = 1$

Set of possible **elementary outcomes**



Specify Likelihood (or probability) of each **elementary outcome**

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Axioms of Probability

Let (Ω, P) be a probability space. Then, the following properties hold for any events $E, F \subseteq \Omega$.

Axiom 1 (Non-negativity): $P(E) \geq 0$.
Axiom 2 (Normalization): $P(\Omega) = 1$.
Axiom 3 (Countable Additivity): If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$

Called "axioms" because all properties of P follow from them!

Corollary 1 (Complementation): $P(E^c) = 1 - P(E)$.
Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$.
Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

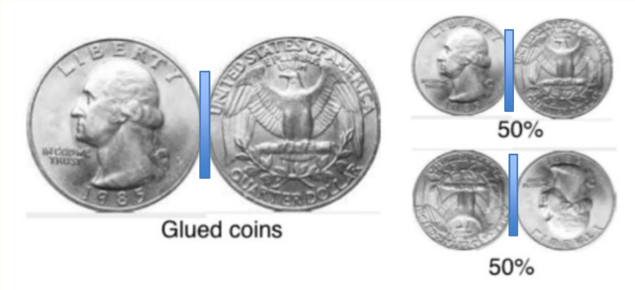
Non-equally Likely Outcomes

Many probability spaces can have **non-equally likely outcomes**.

Biased coin

$P(H) = p$
$P(T) = 1 - p$

Glued coins

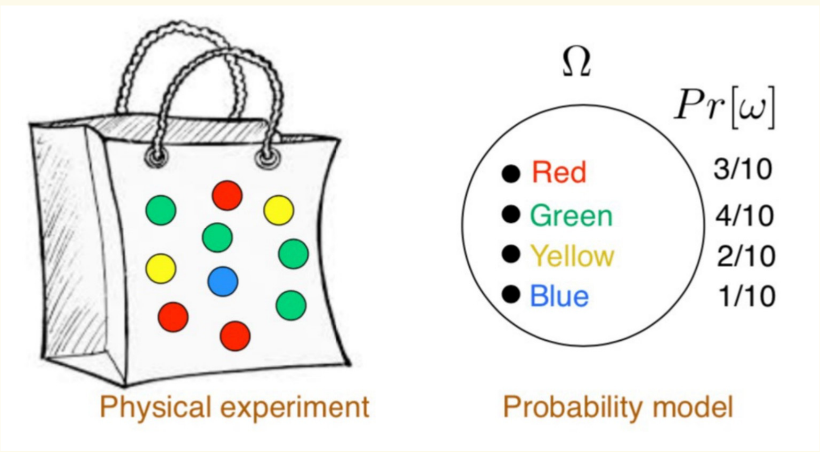


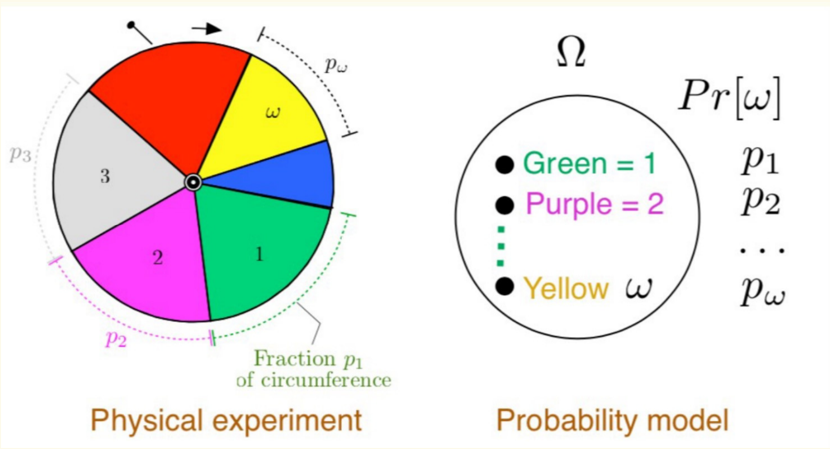
$P(HT) = P(TH) = 0.5$
$P(HH) = P(TT) = 0$

Attached coins

$P(HH) = P(TT) = 0.4$
$P(HT) = P(TH) = 0.1$

More Examples of Non-equally Likely Outcomes





Agenda

- Events
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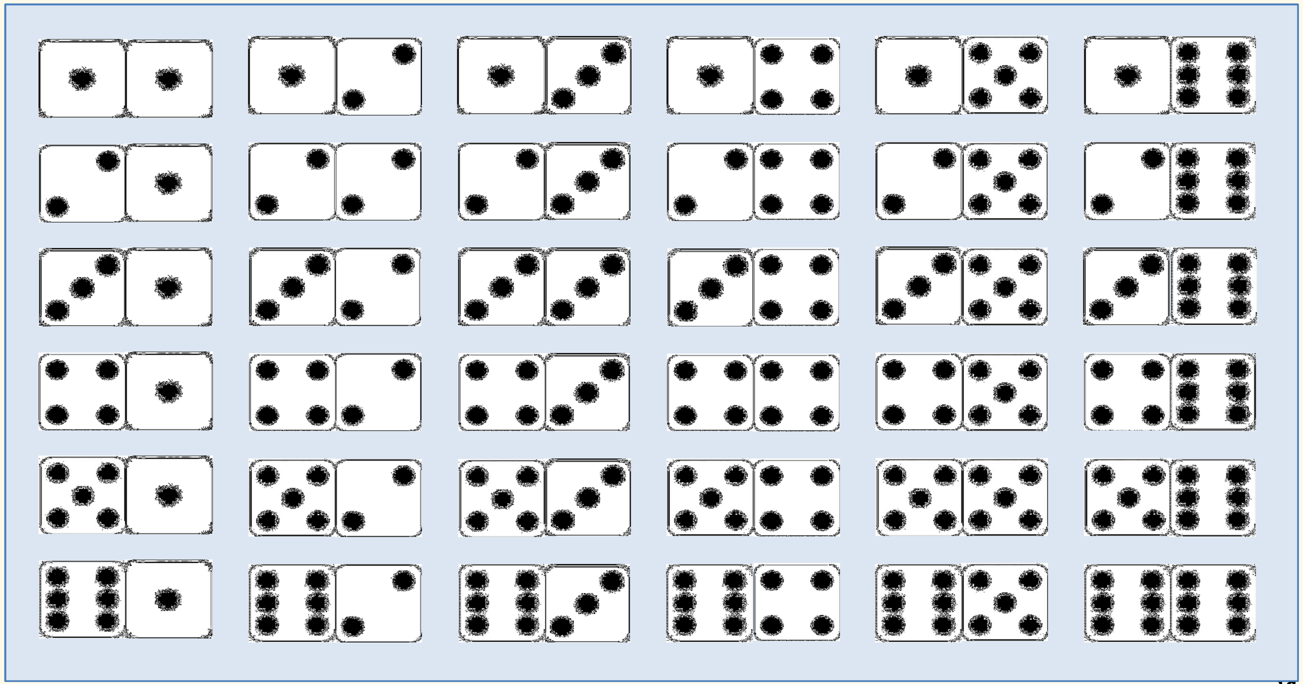
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- Another Example (Equally Likely)

Example: Dice Rolls

Suppose I had two, fair, 6-sided dice that we roll once each.
What is the probability that we see *at least one 3 in the two rolls*?



Example: Dice Rolls

Suppose I had two, fair, 6-sided dice that we roll once each.
What is the probability that we see *at least one 3 in the two rolls*?

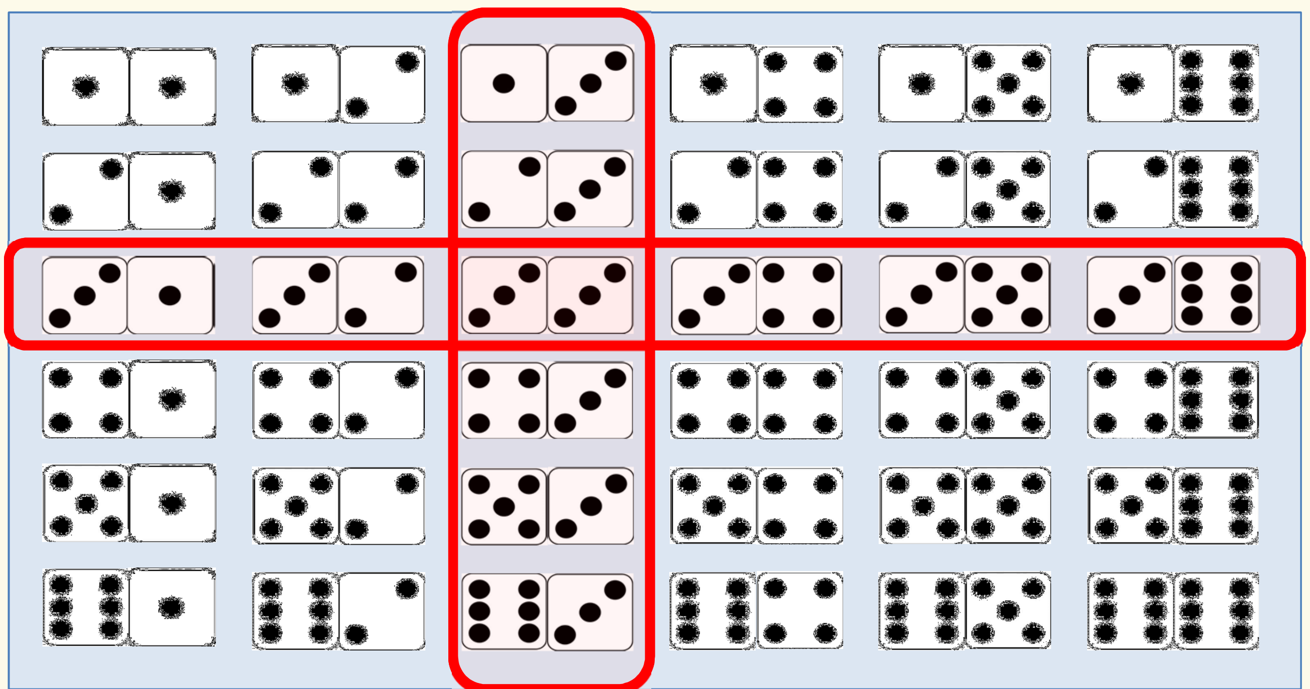
Event has

$$6 + 6 - 1 = 11$$

outcomes

$$|\Omega| = 36$$

$$P(\geq \text{one } 3) = \frac{11}{36}$$



Fun: Von Neumann's Trick with a biased coin

- How to use a biased coin to get a fair coin flip:
  - Suppose that you have a biased coin:
    - $P(H) = p \qquad P(T) = 1 - p$

1. Flip coin twice: If you get HH or TT go to step 1
2. If you got HT output H; if you got TH output T.

Why is it fair? $P(HT) = p(1-p) = (1-p)p = P(TH)$

Drawback: You may never get to step 2.