CSE 312

Foundations of Computing II

Lecture 18: Joint Distributions

Midterm

- Grading is not finished in time for today
  - After grading I will review for fairness before releasing grades
  - With the Friday holiday, grades won't be ready until Monday
  - Solutions will be posted on Canvas pages for Monday

- Please focus on the course content and problem sets while you are waiting…

Agenda

- Joint Distributions
  - Cartesian Products
  - Joint PMFs and Joint Range
  - Marginal Distribution
- Conditional Expectation and Law of Total Expectation

Why joint distributions?

- Given all of its user's ratings for different movies, and any preferences you have expressed, Netflix wants to recommend a new movie for you.
- Given a large amount of medical data correlating symptoms and personal history with diseases, predict what is ailing a person with a particular medical history and set of symptoms.
- Given current traffic, pedestrian locations, weather, lights, etc. decide whether a self-driving car should slow down or come to a stop

Review Cartesian Product

Definition. Let A and B be sets. The **Cartesian product** of A and B is denoted

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Example.

$$\{1,2,3\} \times \{4, 5\} = \{(1, 4), (1,5), (2,4), (2,5), (3, 4), (3,5)\}$$

If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$.

The sets don't need to be finite! You can have $\mathbb{R} \times \mathbb{R}$ (often denoted $\mathbb{R}^2$)

Joint PMFs and Joint Range

Definition. Let X and Y be discrete random variables. The **Joint PMF** of X and Y is

$$p_{X,Y}(a,b) = P(X = a, Y = b)$$

Definition. The **joint range** of $p_{X,Y}$ is

$$\Omega_{X,Y} = \{(c,d) : p_{X,Y}(c,d) > 0\} \subseteq \Omega_X \times \Omega_Y$$

Note that

$$\sum_{(s,t)\in\Omega_{X,Y}} p_{X,Y}(s,t) = 1$$

Example – Weird Dice

Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die.

$\Omega_X = \{1,2,3,4\}$ and $\Omega_Y = \{1,2,3,4\}$

In this problem, the joint PMF is if

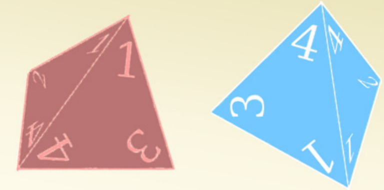
$$p_{X,Y}(x,y) = \begin{cases} 1/16 & \text{if } x, y \in \Omega_{X,Y} \\ 0 & \text{otherwise} \end{cases}$$

X\Y	1	2	3	4
1	1/16	1/16	1/16	1/16
2	1/16	1/16	1/16	1/16
3	1/16	1/16	1/16	1/16
4	1/16	1/16	1/16	1/16

and the joint range is (since all combinations have non-zero probability)

$\Omega_{X,Y} = \Omega_X \times \Omega_Y$

Example – Weirder Dice



Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

$\Omega_U = \{1,2,3,4\}$ and $\Omega_W = \{1,2,3,4\}$

1 $\Rightarrow$ (1, 1)

$\Omega_{U,W} = \{(u, w) \in \Omega_U \times \Omega_W : u \leq w\} \neq \Omega_U \times \Omega_W$

(1,2),(2,1)

Poll: pollev.com/paulbeame028

What is $p_{U,W}(1,3) = P(U = 1, W = 3)$?

a. $1/16$

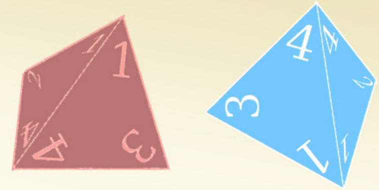
b. $2/16$

c. $1/2$

d. Not sure

U\W	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

Example – Weirder Dice



Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

$\Omega_U = \{1,2,3,4\}$ and $\Omega_W = \{1,2,3,4\}$

$\Omega_{U,W} = \{(u, w) \in \Omega_U \times \Omega_W : u \le w\} \neq \Omega_U \times \Omega_W$

The joint PMF $p_{U,W}(u, w) = P(U = u, W = w)$ is

$$p_{U,W}(u,w) = \begin{cases} 2/16 & \text{if } (u,w) \in \Omega_U \times \Omega_W \text{ where } w > u \\ 1/16 & \text{if } (u,w) \in \Omega_U \times \Omega_W \text{ where } w = u \\ 0 & \text{otherwise} \end{cases}$$

U\W	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

Example – Weirder Dice

Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

Suppose we didn't know how to compute $P(U = u)$ directly. Can we figure it out if we know $p_{U,W}(u, w)$?

Just apply LTP over the possible values of W: $P_U(u)$

$p_U(1) = 7/16$

$p_U(2) = 5/16$

$p_U(3) = 3/16$

$p_U(4) = 1/16$

$\frac{7}{16}$ $\frac{5}{16}$ $\frac{3}{16}$ $\frac{1}{16}$

U\W	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

Marginal PMF

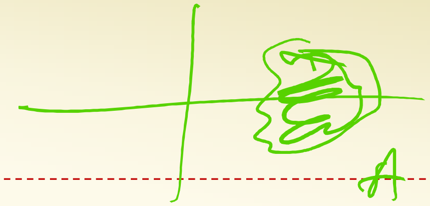
Definition. Let X and Y be discrete random variables and $p_{X,Y}(a,b)$ their joint PMF. The **marginal PMF** of X

$$p_X(a) = \sum_{b \in \Omega_Y} p_{X,Y}(a,b)$$

$$p_Y(b) = \sum_{a \in \Omega_X} p_{X,Y}(a,b)$$

Similarly, $p_Y(b) = \sum_{a \in \Omega_X} p_{X,Y}(a,b)$

Continuous distributions on $\mathbb{R} \times \mathbb{R}$



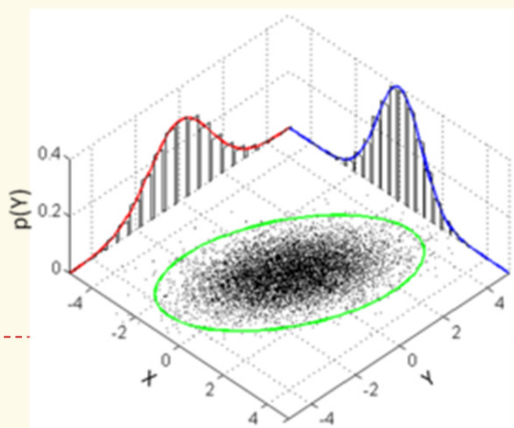
Definition. The **joint probability density function (PDF)** of continuous random variables X and Y is a function $f_{X,Y}$ defined on $\mathbb{R} \times \mathbb{R}$ such that

- $f_{X,Y}(x, y) \geq 0$ for all $x, y \in \mathbb{R}$
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \mathrm{d}x \, \mathrm{d}y = 1$

for $\boxed{A \subseteq \mathbb{R} \times \mathbb{R}}$ the probability that $(X, Y) \in A$ is $\iint_A f_{X,Y}(x, y) \, \mathrm{d}x\mathrm{d}y$

The **(marginal) PDFs** f_X and f_Y are given by

- $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, \mathrm{d}y$
- $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, \mathrm{d}x$



Independence and joint distributions

Definition. Discrete random variables X and Y are **independent** iff

- $p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y)$ for all $x \in \Omega_X, y \in \Omega_Y$

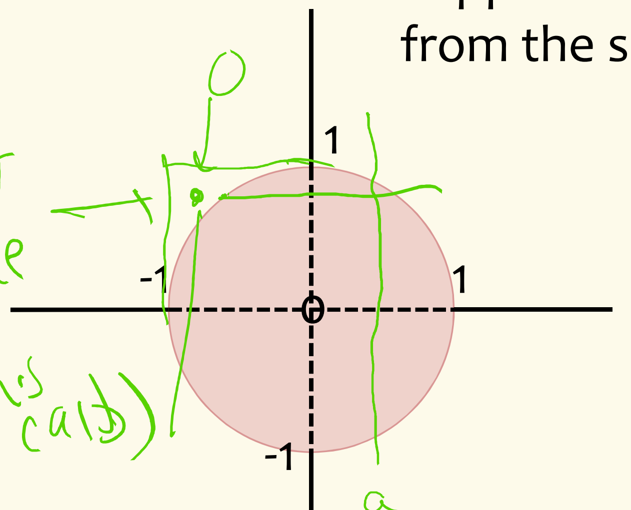
$P(X=x, Y=y)$ $\quad$ $P(X=x) \cdot P(Y=y)$

Definition. Continuous random variables X and Y are **independent** iff

- $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$ for all $x, y \in \mathbb{R}$

Example – Uniform distribution on a unit disk

Suppose that a pair of random variabes (X, Y) is chosen uniformly from the set of real points (x, y) such that $x^2 + y^2 \leq 1$



not true at this (a,b)

$f_{X,Y}(a,b) = f_X(a) f_Y(b)$

This is a disk of radius 1 which has area π

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} \mathrm{d}y$$

$$= 2\sqrt{1 - x^2}/\pi$$

Poll: pollev.com/paulbeame028

Are X and Y independent?

a. Yes

b. No

Joint Expectation

$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

Definition. Let X and Y be discrete random variables and $p_{X,Y}(a,b)$ their joint PMF. The **expectation** of some function $g(x,y)$ with inputs X and Y

$$\mathbb{E}[g(X,Y)] = \sum_{a \in \Omega_X} \sum_{b \in \Omega_Y} g(a,b) \cdot p_{X,Y}(a,b)$$

Later

$$= \sum_{(a,b) \in \Omega_{X,Y}}$$

Brain Break

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- Joint Distributions
  - Cartesian Products
  - Joint PMFs and Joint Range
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- Conditional Expectation and Law of Total Expectation

Conditional Expectation

Definition. Let X be a discrete random variable then the **conditional expectation** of X given event A is

$$\mathbb{E}[X \mid A] = \sum_{x \in \Omega_X} x \cdot P(X = x \mid A)$$

Notes:

- Can be phrased as a "random variable version"

$$\mathbb{E}[X \mid Y = y]$$

- Linearity of expectation still applies here

$$\mathbb{E}[aX + bY + c \mid A] = a\,\mathbb{E}[X \mid A] + b\,\mathbb{E}[Y \mid A] + c$$

Law of Total Expectation

Law of Total Expectation (event version). Let X be a random variable and let events $A_1, \dots, A_n$ partition the sample space. Then,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i] \cdot P(A_i)$$

Law of Total Expectation (random variable version). Let X be a random variable and Y be a discrete random variable. Then,

$$\mathbb{E}[X] = \sum_{y \in \Omega_Y} \mathbb{E}[X \mid Y = y] \cdot P(Y = y)$$

Proof of Law of Total Expectation

Follows from Law of Total Probability and manipulating sums

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x)$$

$$= \sum_{x \in \Omega_X} x \cdot \sum_{i=1}^{n} P(X = x \mid A_i) \cdot P(A_i)$$ (by LTP)

$$= \sum_{i=1}^{n} P(A_i) \sum_{x \in \Omega_X} x \cdot P(X = x \mid A_i)$$ (change order of sums)

$$= \sum_{i=1}^{n} P(A_i) \cdot \mathbb{E}[X \mid A_i]$$ (def of cond. expect.)

Example – Flipping a Random Number of Coins

Suppose someone gave us $Y \sim \text{Poi}(5)$ fair coins and we wanted to compute the expected number of heads X from flipping those coins.

By the Law of Total Expectation

$$\mathbb{E}[X] = \sum_{i=0}^{\infty} \mathbb{E}[X \mid Y = i] \cdot P(Y = i) = \sum_{i=0}^{\infty} \frac{i}{2} \cdot P(Y = i)$$

$$= \frac{1}{2} \cdot \sum_{i=0}^{\infty} i \cdot P(Y = i) \quad = \mathbb{E}[Y]$$

$$= \frac{1}{2} \cdot \mathbb{E}[Y] = \frac{1}{2} \cdot 5 = 2.5$$