CSE 312

# Foundations of Computing II

## Lecture 22: Maximum Likelihood Estimation (MLE)

my OH today
2:30–4:20
Also on Zoom
See link in Edstem/General!

# Agenda

- Idea: Estimation
- Maximum Likelihood Estimation (example: mystery coin)
- Continuous MLE

# Recap Probability vs Statistics

$\text{Ber}(p = 0.5)$ → **Probability** Given model, predict data → $P(THHTHH)$

$\text{Ber}(p = ??)$ ← **Statistics** Given data, predict model ← $THHTHH$

# Recap Formalizing Polls

We assume that poll answers $X_1, \ldots, X_n \sim \mathrm{Ber}(p)$ i.i.d. for unknown $p$

**Goal:** Estimate $p$

We did this by computing $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} X_i$

## Recap More generally …

In estimation we… .

- Assume: we know the type of the random variable that we are observing independent samples from
  - We just don't know the parameters, e.g.
    - the bias $p$ of a random coin $\text{Bernoulli}(p)$
    - The arrival rate $\lambda$ for the $\text{Poisson}(\lambda)$ or $\text{Exponential}(\lambda)$
    - The mean $\mu$ and variance $\sigma$ of a normal $\mathcal{N}(\mu, \sigma)$

- Goal: find the "best" parameters to fit the data

## Notation – Parametric Model (discrete case)

**Definition.** A **(parametric) model** is a family of distributions indexed by a parameter $\theta$, described by a two-argument function

$P(x;\theta) =$ prob. of outcome $x$ when distribution has parameter $\theta$

[illegible]

[i.e., every $\theta$ defines a different distribution $\sum_x P(x;\theta) = 1$]

**Examples**

- "Bernoullis": $P(x;\theta = p) = \begin{cases} p & x = 1 \\ 1 - p & x = 0 \end{cases}$
- "Geometrics": $P(i;\theta = p) = (1-p)^{i-1}p$ for $i \in \mathbb{N}$

$i \in \{0, 1, 2, \dots\} = \mathbb{N}$

# Statistics: Parameter Estimation – Workflow

Parameter estimate

**Distribution** $P(x;\theta)$ → Independent samples $X_1, \dots, X_n$ from $P(x;\theta)$ → **Estimation Algorithm** → $\hat{\theta}$

fixed

"$\theta$ hat"

$\theta$ = unknown parameter

**Example:** coin flip distribution with unknown $\theta$ = probability of heads

Observation: $HTTHHHTHTHTTTTHTHTTTTTHT$

**Goal:** Estimate $\theta$

# Example

Suppose we have a mystery coin with some probability $p$ of coming up heads. We flip the coin 8 times, independent of other flips, and see the following sequence flips

$$TTHTHTTH$$

Given this data, what would you estimate $p$ is?

3 heads out of 8 tosses

Poll: pollev.com/paulbeame028

a. 1/2
b. 5/8
c. 3/8
d. 1/4

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# Likelihood

4/5

Say we see outcome $HHTHH$.

You tell me your best guess about the value of the unknown parameter $\theta$ (a.k.a. $p$) is 4/5. Is there some way that you can argue "objectively" that this is the best estimate?

# Likelihood

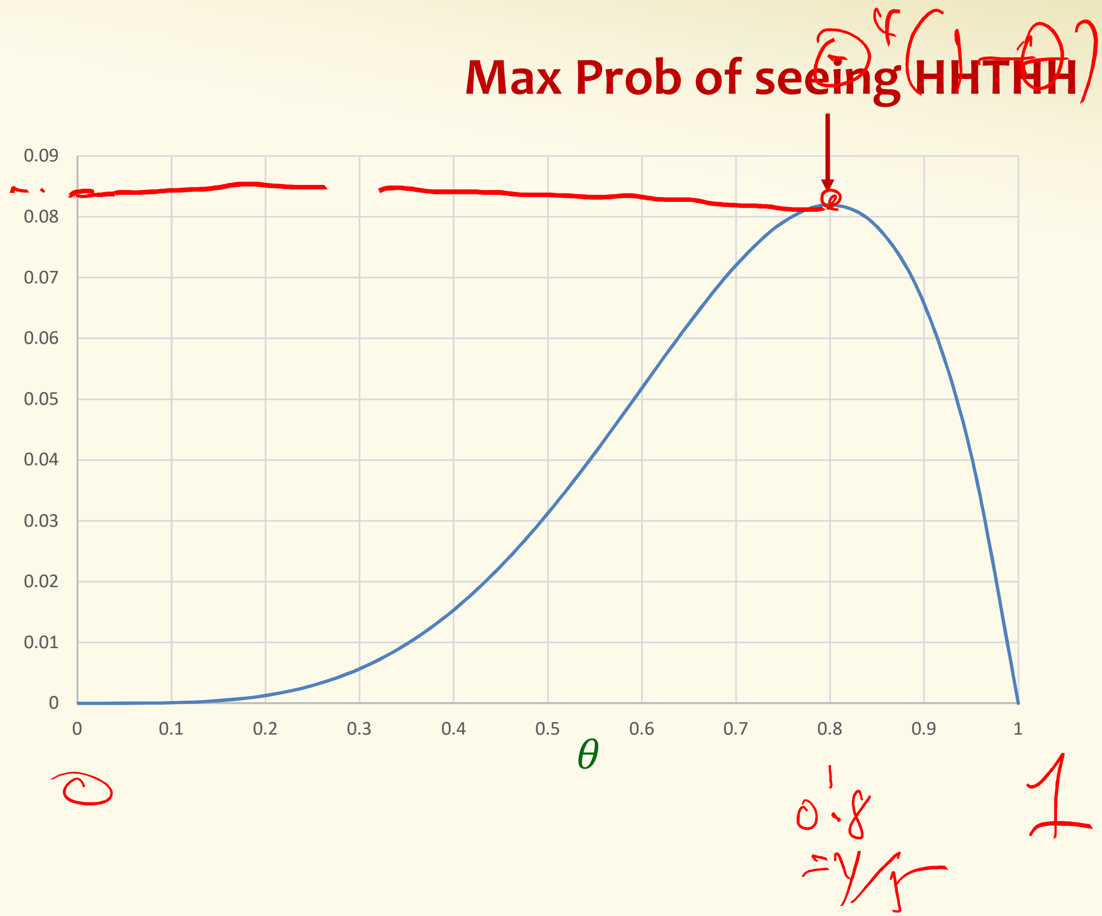
Say we see outcome $HHTHH$.

$$\mathcal{L}(HHTHH \mid \theta) = \theta^4(1-\theta)$$

Probability of observing the outcome $HHTHH$ if $\theta =$ prob. of heads.

For a fixed outcome $HHTHH$ , this is a function of $\theta$.

\mathcal(L)



## Likelihood of Different Observations

(Discrete case)

**Definition.** The **likelihood** of independent observations $x_1, \dots, x_n$ is

$$\mathcal{L}(x_1, \dots, x_n \mid \theta) = \prod_{i=1}^{n} P(x_i; \theta)$$

**Maximum Likelihood Estimation (MLE).** Given data $x_1, \dots, x_n$, find $\hat{\theta}$ such that $\mathcal{L}(x_1, \dots, x_n \mid \hat{\theta})$ is maximized!

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\ \mathcal{L}(x_1, \dots, x_n \mid \theta)$$

Usually: Solve $\boxed{\frac{\partial \mathcal{L}(x_1, \dots, x_n \mid \theta)}{\partial \theta} = 0}$ or $\frac{\partial \ln \mathcal{L}(x_1, \dots, x_n \mid \theta)}{\partial \theta} = 0$ [+check it's a max!]

ln log base e

## Likelihood vs. Probability

- Fixed $\theta$: **probability** $\prod_{i=1}^{n} P(x_i;\theta)$ that dataset $x_1, \dots, x_n$ is sampled by distribution with parameter $\theta$
  - A function of $x_1, \dots, x_n$
- Fixed $x_1, \dots, x_n$: **likelihood** $\mathcal{L}(x_1, \dots, x_n \mid \theta)$ that parameter $\theta$ explains dataset $x_1, \dots, x_n$.
  - A function of $\theta$

These notions are the same number if we fix *both* $x_1, \dots, x_n$ and $\theta$, but different role/interpretation

## Example – Coin Flips

Specific string

Observe: Coin-flip outcomes $x_1, \ldots, x_n$, with $n_H$ heads, $n_T$ tails

– i.e., $n_H + n_T = n$

**Goal:** estimate $\theta$ = prob. heads.

$$\mathcal{L}(x_1, \ldots, x_n | \theta) = \theta^{n_H} (1 - \theta)^{n_T}$$

$$\frac{\partial}{\partial \theta} \mathcal{L}(x_1, \ldots, x_n | \theta) = ???$$

While it is possible to compute this derivative, it's not always nice since we are working with products.

# Log-Likelihood

$$\frac{d \ln(x)}{dx} = \frac{1}{x}$$

We can save some work if we use the **log-likelihood** instead of the likelihood directly.

**Definition.** The **log-likelihood** of independent observations $x_1, \dots, x_n$ is

$$\ln \mathcal{L}(x_1, \dots, x_n \mid \theta) = \ln \prod_{i=1}^{n} P(x_i; \theta) = \sum_{i=1}^{n} \ln P(x_i; \theta)$$

Useful log properties

$$\ln(ab) = \ln(a) + \ln(b)$$
$$\ln(a/b) = \ln(a) - \ln(b)$$
$$\ln\left(a^b\right) = b \cdot \ln(a)$$

# Example – Coin Flips

$\frac{n_H}{\hat{\theta}} = \frac{n_T}{1-\hat{\theta}} \Rightarrow (1-\hat{\theta})n_H = \hat{\theta} n_T$

$\Rightarrow n_H = \hat{\theta}(n_H + n_T)$

$\therefore n_H = \hat{\theta} n$

Observe: Coin-flip outcomes $x_1, \ldots, x_n$, with $n_H$ heads, $n_T$ tails

– i.e., $n_H + n_T = n$

**Goal:** estimate $\theta$ = prob. heads.

$$\mathcal{L}(x_1, \ldots, x_n | \theta) = \theta^{n_H} (1-\theta)^{n_T}$$

$$\ln \mathcal{L}(x_1, \ldots, x_n | \theta) = n_H \ln \theta + n_T \ln(1-\theta)$$

$$\frac{\partial}{\partial \theta} \ln \mathcal{L}(x_1, \ldots, x_n | \theta) = n_H \cdot \frac{1}{\theta} - n_T \cdot \frac{1}{1-\theta} = 0$$

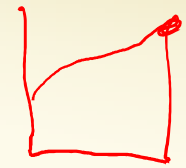
Want value $\hat{\theta}$ of $\theta$ s.t. $\frac{\partial}{\partial \theta} \ln \mathcal{L}(x_1, \ldots, x_n | \theta) = 0$

So we need $n_H \cdot \frac{1}{\hat{\theta}} - n_T \cdot \frac{1}{1-\hat{\theta}} = 0$ $\boxed{\hat{\theta} = \frac{n_H}{n}}$

Solving gives

$$\hat{\theta} = \frac{n_H}{n}$$

## General Recipe



1. **Input** Given $n$ i.i.d. samples $x_1, \ldots, x_n$ from parametric model with parameter $\theta$.
2. **Likelihood** Define your likelihood $\mathcal{L}(x_1, \ldots, x_n \mid \theta)$.
   – For discrete $\mathcal{L}(x_1, \ldots, x_n \mid \theta) = \prod_{i=1}^{n} P(x_i \,; \theta)$
3. **Log** Compute $\ln \mathcal{L}(x_1, \ldots, x_n \mid \theta)$
4. **Differentiate** Compute $\frac{\partial}{\partial \theta} \ln \mathcal{L}(x_1, \ldots, x_n \mid \theta)$
5. **Solve for** $\hat{\theta}$ by setting derivative to $0$ and solving for max.

Generally, you need to do a second derivative test to verify it is a maximum, but we won't ask you to do that in CSE 312.

# Brain Break

$$\frac{n_H}{\hat{\theta}} = \frac{n_T}{1-\hat{\theta}}$$

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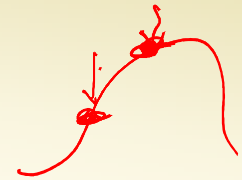
## The Continuous Case



PDF

$\theta$ fixed

Given $n$ (independent) samples $x_1, \ldots, x_n$ from (continuous) parametric model $f(x_i; \theta)$ which is now a family of densities

**Definition.** The **likelihood** of independent observations $x_1, \ldots, x_n$ is

$$\mathcal{L}(x_1, \ldots, x_n \mid \theta) = \prod_{i=1}^{n} f(x_i; \theta)$$

Density function! (Why?)

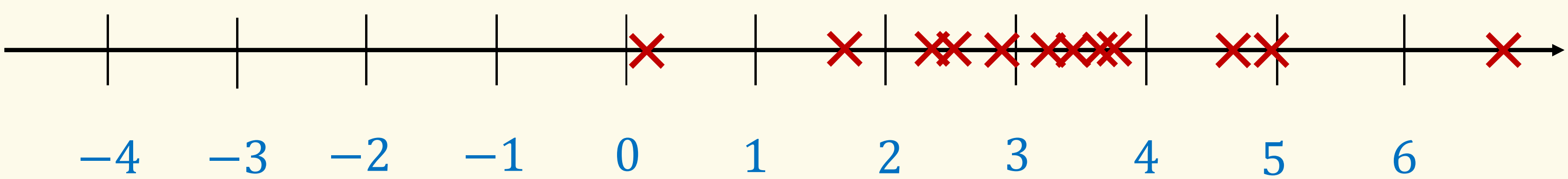
## Why density?

- Density ≠ probability, but:
  - For maximizing likelihood, we really only care about relative likelihoods, and density captures that
  - has desired property that likelihood increases with better fit to the model

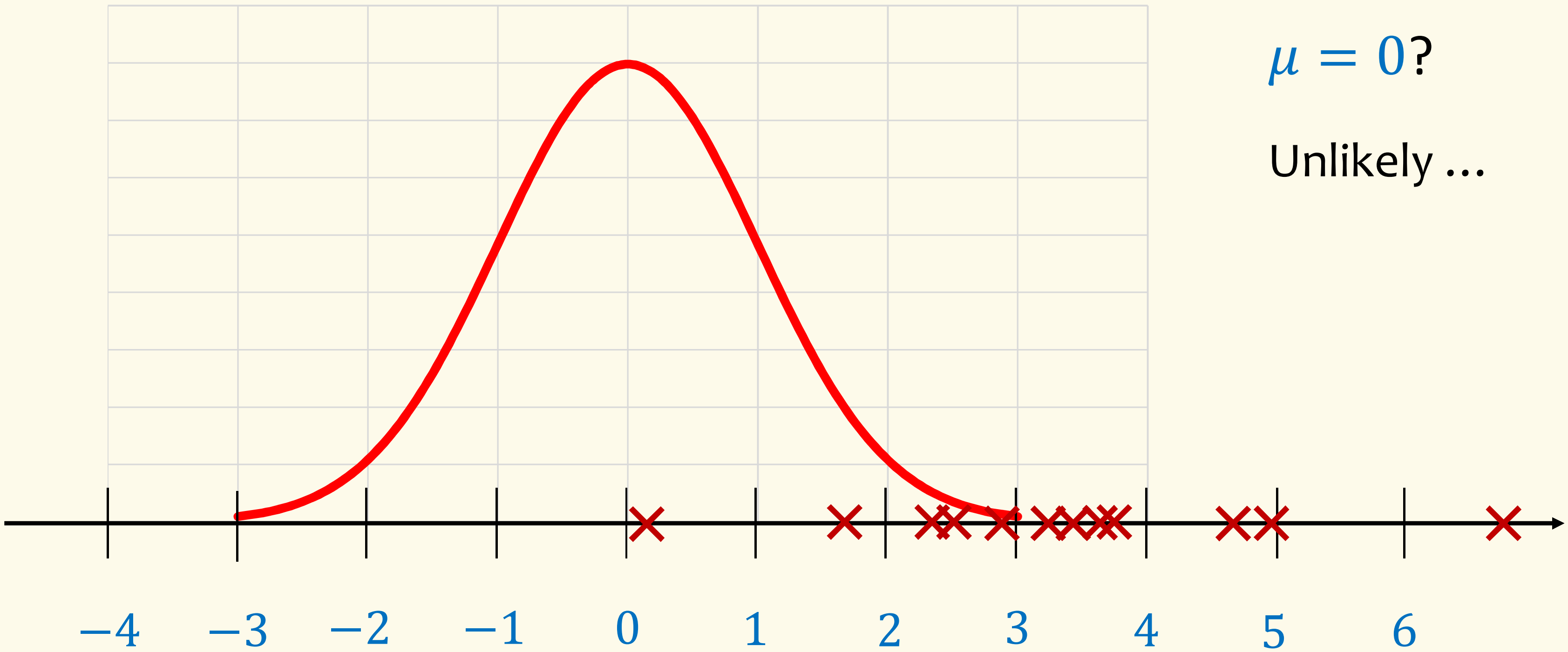
$\sigma = 1$

$n$ samples $x_1, \dots, x_n \in \mathbb{R}$ from Gaussian $\mathcal{N}(\mu, 1)$. <u>Most likely</u> $\mu$?

[i.e., we are given the <u>promise</u> that the variance is 1]



$n$ samples $x_1, \ldots, x_n \in \mathbb{R}$ from Gaussian $\mathcal{N}(\mu, 1)$. Most likely $\mu$?



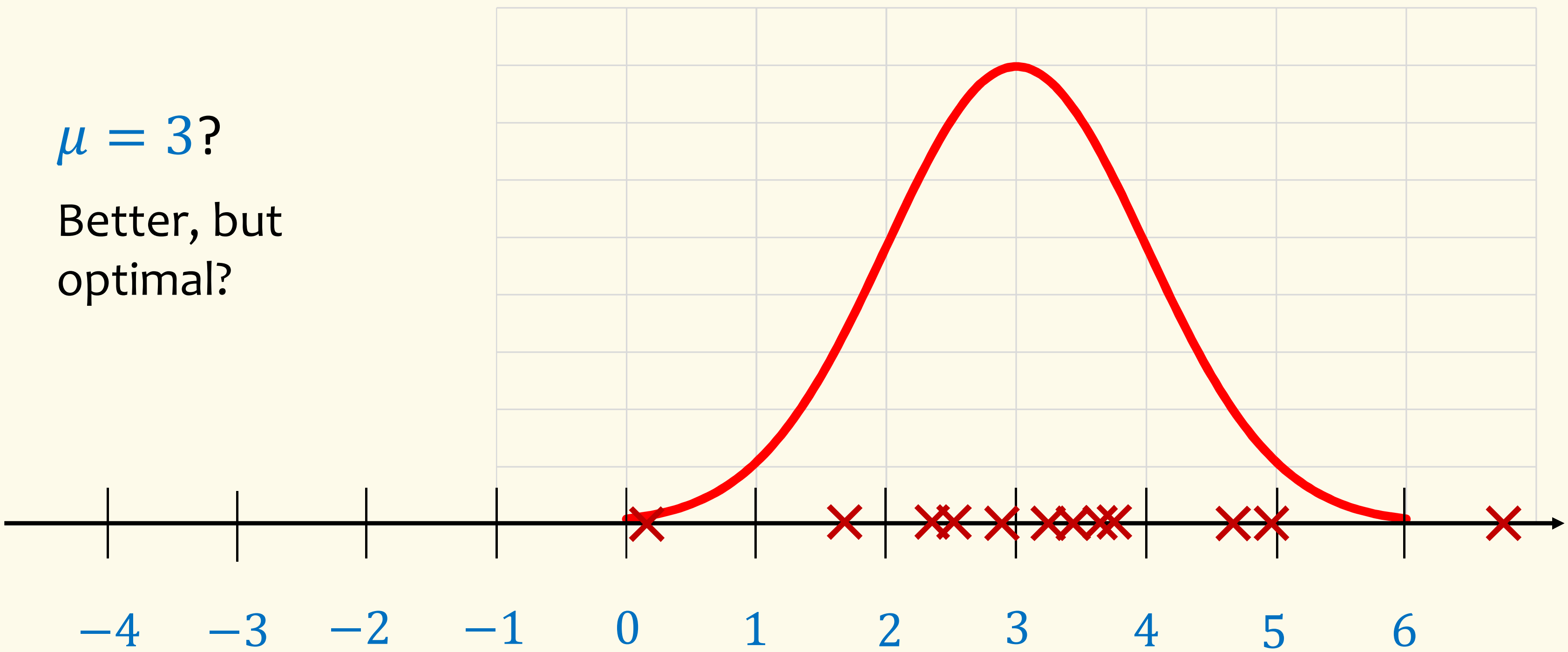

$\mu = 0$?

Unlikely …

$n$ samples $x_1, \ldots, x_n \in \mathbb{R}$ from Gaussian $\mathcal{N}(\mu, 1)$. Most likely $\mu$?

$\mu = 3$?

Better, but optimal?



## Example – Gaussian Parameters

$f(x_i, \mu)\ \frac{1}{\sqrt{2\pi\sigma}} e^{-(x_i-\mu)^2}$

Normal outcomes $x_1, \dots, x_n$, known variance $\sigma^2 = 1$ but *unknown* mean $\mu$

$$\boxed{f(x_i, \mu) = \frac{1}{\sqrt{2\pi}} e^{-(x_i - \mu)^2}}$$

**Goal:** estimate $\theta$ = mean

**Next time:**

$$\hat{\theta} = \frac{\sum_i^n x_i}{n}$$

In other words, MLE is the *sample mean* of the data.

# General Recipe

1. **Input** Given $n$ i.i.d. samples $x_1, \dots, x_n$ from parametric model with parameter $\theta$.
2. **Likelihood** Define your likelihood $\mathcal{L}(x_1, \dots, x_n \mid \theta)$.
   - For discrete $\mathcal{L}(x_1, \dots, x_n \mid \theta) = \prod_{i=1}^{n} P(x_i\,;\theta)$
   - For continuous $\mathcal{L}(x_1, \dots, x_n \mid \theta) = \prod_{i=1}^{n} f(x_i\,;\theta)$
3. **Log** Compute $\ln \mathcal{L}(x_1, \dots, x_n \mid \theta)$
4. **Differentiate** Compute $\frac{\partial}{\partial\theta} \ln \mathcal{L}(x_1, \dots, x_n \mid \theta)$
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