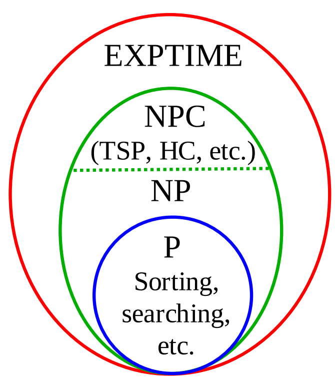
Lecture 25: AlgoRhythm Design Techniques

- Agenda for today's class:
  - Coping with NP-complete and other hard problems
    - Approximation using Greedy Techniques
      - Optimally bagging groceries: Bin Packing
    - Divide & Conquer Algorithms and their Recurrences
    - Dynamic Programming by "memoizing"
      - Fibonacci's Revenge
    - Randomized Data Structures and Algorithms
      - Treaps
      - "Probably correct" primality testing
    - In the Sections on Thursday: Backtracking
      - Game Trees, minimax, and alpha-beta pruning
- Read Chapter 10 and Sec 12.5 in the textbook

Recall: P, NP, and Exponential Time Problems

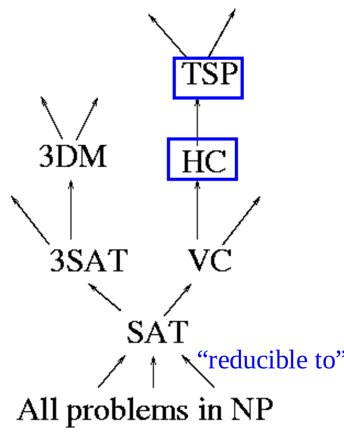
- Diagram depicts relationship between P, NP, and EXPTIME (class of problems that can be solved within exponential time)
- NP-Complete problem = problem in NP to which all other NP problems can be reduced
  - Can convert input for a given NP problem to input for NPC problem
- All algorithms for NP-C problems so far have tended to run in nearly exponential worst case time



It is believed that
$P \neq NP \neq EXPTIME$

The “Curse” of NP-completeness

- Cook first showed (in 1971) that satisfiability of Boolean formulas (SAT) is NP-Complete
- Hundreds of other problems (from scheduling and databases to optimization theory) have since been shown to be NPC
- **No polynomial time algorithm is known for any NPC problem!**



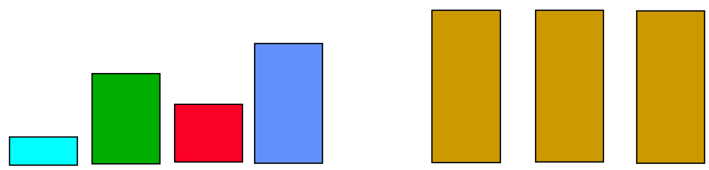
Coping strategy #1: Greedy Approximations

- Use a greedy algorithm to solve the given problem
  - Repeat until a solution is found:
    - Among the set of possible next steps:
      Choose the current best-looking alternative and commit to it
- Usually fast and simple
- Works in some cases…(always finds optimal solutions)
  - Dijsktra’s single-source shortest path algorithm
  - Prim’s and Kruskal’s algorithm for finding MSTs
- but not in others…(may find an approximate solution)
  - TSP – always choosing current least edge-cost node to visit next
  - Bagging groceries…

The Grocery Bagging Problem

- You are an environmentally-conscious grocery bagger at QFC
- You would like to minimize the total number of bags needed to pack each customer's items.

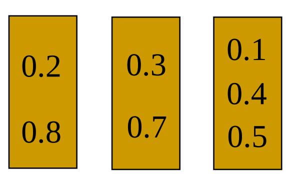
Items (mostly junk food) Grocery bags



Sizes $s_1, s_2, \ldots, s_N$ $(0 < s_i \leq 1)$ Size of each bag = 1

Optimal Grocery Bagging: An Example

- Example: Items = 0.5, 0.2, 0.7, 0.8, 0.4, 0.1, 0.3
  - How may bags of size 1 are required?



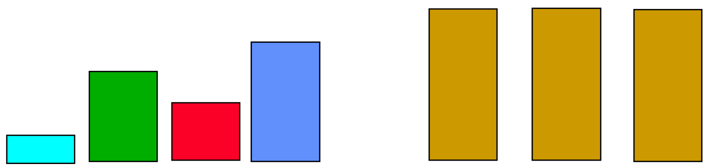
Only 3 bags required

- Can find optimal solution through exhaustive search
  - Search all combinations of N items using 1 bag, 2 bags, etc.
  - Takes **exponential time!**

Bagging groceries is NP-complete

- **Bin Packing problem**: Given N items of sizes $s_1, s_2, \ldots, s_N$ ($0 < s_i \le 1$), pack these items in the least number of bins of size 1.

Items Bins



Sizes $s_1, s_2, \ldots, s_N$ ($0 < s_i \le 1$) Size of each bin = 1

- The general bin packing problem is NP-complete
  - Reductions: All NP-problems → SAT → 3SAT → 3DM → PARTITION → Bin Packing (see Garey & Johnson, 1979)

Greedy Grocery Bagging

- Greedy strategy #1 "**First Fit**":
  1. Place each item in first bin large enough to hold it
  2. If no such bin exists, get a new bin
- **Example**: Items = 0.5, 0.2, 0.7, 0.8, 0.4, 0.1, 0.3

Greedy Grocery Bagging

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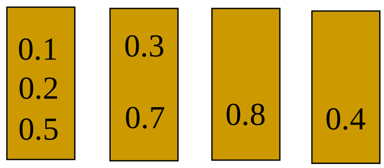


Uses 4 bins
Not optimal

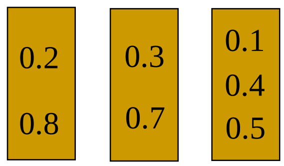
- **Approximation Result**: If M is the optimal number of bins, First Fit never uses more than $\lceil \mathbf{1.7M} \rceil$ bins (see textbook).

Getting Better at Greedy Grocery Bagging

- Greedy strategy #2 "First Fit Decreasing":
  1. Sort items according to *decreasing size*
  2. Place each item in first bin large enough to hold it
- **Example**: Items = 0.5, 0.2, 0.7, 0.8, 0.4, 0.1, 0.3

Getting Better at Greedy Grocery Bagging

- Greedy strategy #2 "First Fit Decreasing":
  1. Sort items according to *decreasing size*
  2. Place each item in first bin large enough to hold it
- **Example**: Items = 0.5, 0.2, 0.7, 0.8, 0.4, 0.1, 0.3



Uses 3 bins
Optimal in this case
Not optimal in general

- **Approximation Result**: If M is the optimal number of bins, First Fit Decreasing never uses more than **1.2M + 4** bins (see textbook).

Coping Stategy #2: Divide and Conquer

- Basic Idea:
  1. Divide problem into multiple smaller parts
  2. Solve smaller parts ("divide")
     - Solve base cases directly
     - Solve non-base cases recursively
  3. Merge solutions of smaller parts ("conquer")
- Elegant and simple to implement
  - E.g. Mergesort, Quicksort, etc.
- Run time T(N) analyzed using a recurrence relation:
  - $T(N) = aT(N/b) + \Theta(N^k)$ where $a \geq 1$ and $b > 1$

No. of parts (a) — Part size (N/b) — Time for merging solutions ($\Theta(N^k)$)

Analyzing Divide and Conquer Algorithms

- Run time T(N) analyzed using a recurrence relation:
  - $T(N) = aT(N/b) + \Theta(N^k)$ where $a \geq 1$ and $b > 1$
- General solution (see theorem 10.6 in text):

$$T(N) = \begin{cases} O(N^{\log_b a}) & \text{if } a > b^k \\ O(N^k \log N) & \text{if } a = b^k \\ O(N^k) & \text{if } a < b^k \end{cases}$$

- Examples:
  - Mergesort: $a = b = 2$, $k = 1 \rightarrow T(N) = O(N \log N)$
  - Three parts of half size and k = 1 $\rightarrow T(N) = O(N^{\log_2 3}) = O(N^{1.59})$
  - Three parts of half size and k = 2 $\rightarrow T(N) = O(N^2)$

Another Example of D & C

- Recall our old friend Signor Fibonacci and his numbers:

1, 1, 2, 3, 5, 8, 13, 21, 34, …

Leonardo Pisano
Fibonacci (1170-1250)

- First two are: $F_0 = F_1 = 1$
- Rest are sum of preceding two
- $F_n = F_{n-1} + F_{n-2}$ (n > 1)

A D & C Algorithm for Fibonacci Numbers

```
public static int fib(int i) {
  if (i < 0) return 0;  //invalid input
  if (i == 0 || i == 1) return 1;  //base cases
  else return fib(i-1)+fib(i-2);
  }
```

- Easy to write: looks like the definition of F_n
- But what is the running time T(N)?

Recursive Fibonacci

```
public static int fib(int N) {
  if (N < 0) return 0;  // time = 1 for the < operation
  if (N == 0 || N == 1) return 1;  // time = 3 for 2 ==, 1 ||
  else return fib(N-1)+fib(N-2); // T(N-1)+T(N-2)+1
  }
```

- Running time $T(N) = T(N-1) + T(N-2) + 5$
- Using $F_n = F_{n-1} + F_{n-2}$ we can show by induction that $T(N) \geq F_N$.
- We can also show by induction that $F_N \geq (3/2)^N$

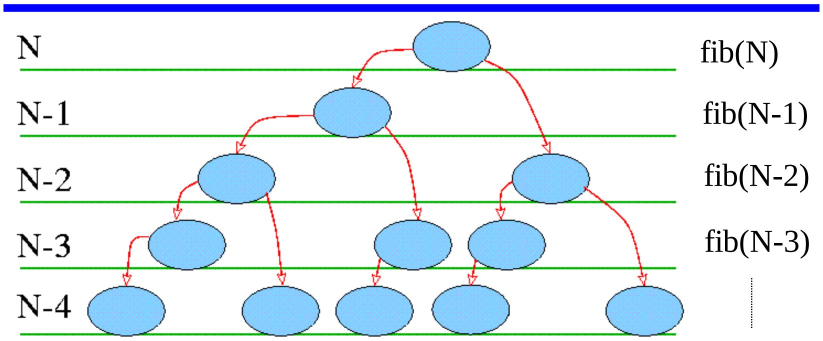
Recursive Fibonacci

- ```
  public static int fib(int N) {
    if (N < 0) return 0;  // time = 1 for the < operation
    if (N == 0 || N == 1) return 1;  // time = 3 for 2 ==, 1 ||
    else return fib(N-1)+fib(N-2); // T(N-1)+T(N-2)+1
  }
  ```
- Running time $T(N) = T(N-1) + T(N-2) + 5$
- Therefore, $T(N) \geq (3/2)^N$

  i.e. $T(N) = \Omega((1.5)^N)$

Yikes…exponential running time!

The Problem with Recursive Fibonacci



- Wastes precious time by **re-computing fib(N-i) over and over again**, for i = 2, 3, 4, etc.!
  ```

Solution: "Memoizing" (Dynamic Programming)

- Basic Idea: Use a table to store subproblem solutions
  - Compute solution to a subproblem only once
  - Next time the solution is needed, just look-up the table
- General Structure of DP algorithms:
  - Define problem in terms of smaller subproblems
  - Solve & record solution for each subproblem & base cases
  - Build solution up from solutions to subproblems

Memoized (DP-based) Fibonacci

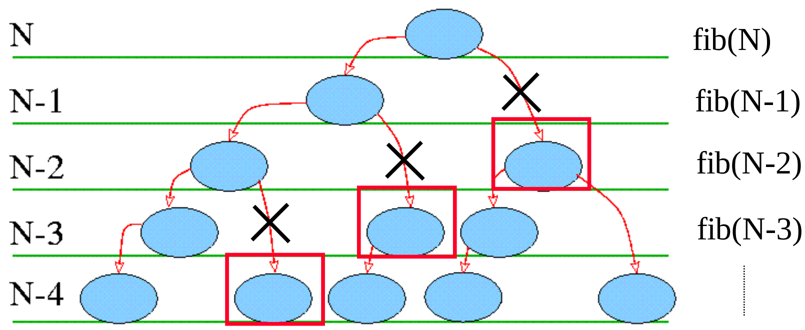
```
public static int fib(int i) {
 // create a global array fibs to hold fib numbers
 // int fibs[N];  // Initialize array fibs to 0's

  if (i < 0) return 0;  //invalid input
  if (i == 0 || i == 1) return 1;  //base cases
  // compute value only if previously not computed
  if (fibs[i] == 0)
      fibs[i] = fib(i-1)+fib(i-2);  //update table (memoize!)

  return fibs[i];
  }
```

Run Time = ?

The Power of DP



- Each value computed only once! No multiple recursive calls
- N values needed to compute fib(N) Run Time = O(N)

Summary of Dynamic Programming

- Very important technique in CS: Improves the run time of D & C algorithms whenever there are shared subproblems
- Examples:
  - DP-based Fibonacci
  - Ordering matrix multiplications
  - Building optimal binary search trees
  - All-pairs shortest path
  - DNA sequence alignment
  - Optimal action-selection and reinforcement learning in robotics
  - etc.

Coping Strategy #3: Viva Las Vegas! (Randomization)

- Basic Idea: When faced with several alternatives, toss a coin and make a decision
  - Utilizes a pseudorandom number generator (Sec. 10.4.1 in text)
- Example: Randomized QuickSort
  - Choose pivot randomly among array elements
- Compared to choosing first element as pivot:
  - Worst case run time is $O(N^2)$ in both cases
    - Occurs if largest chosen as pivot at each stage
  - BUT: For same input, randomized algorithm most likely won't repeat bad performance whereas deterministic quicksort will!
  - Expected run time for randomized quicksort is O(N log N) time for *any* input

Randomized Data Structures

- We've seen many data structures with good average case performance on random inputs, but bad behavior on particular inputs
  - E.g. Binary Search Trees
- Instead of randomizing the input (which we cannot!), consider randomizing the data structure!

What's the Difference?

- Deterministic data structure with good average time over all inputs
  - If your application happens to always contain the "bad" inputs, you are in big trouble!
- Randomized data structure with good expected time for any input
  - Once in a while you will have an expensive operation, but no input can make this happen all the time
- *Kind of like an insurance policy for your algorithm!*

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- *Kind of like an insurance policy for your algorithm!*

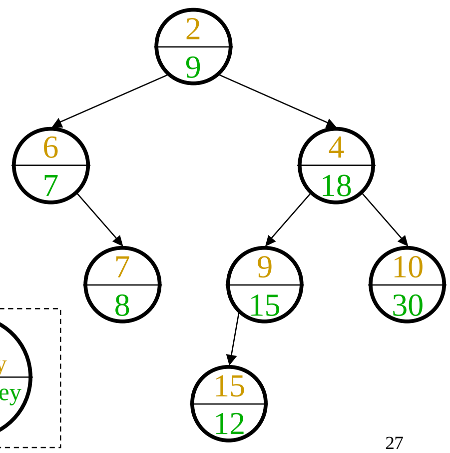
Allstate
You're in good hands.

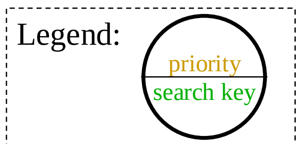
(Disclaimer: Allstate wants nothing to do with this boring lecture or lecturer.)

Example: Treaps (= Trees + Heaps)

- Treaps have both the binary search tree property as well as the heap-order property
- Two keys at each node
  - Key 1 = search element
  - Key 2 = **randomly assigned priority**

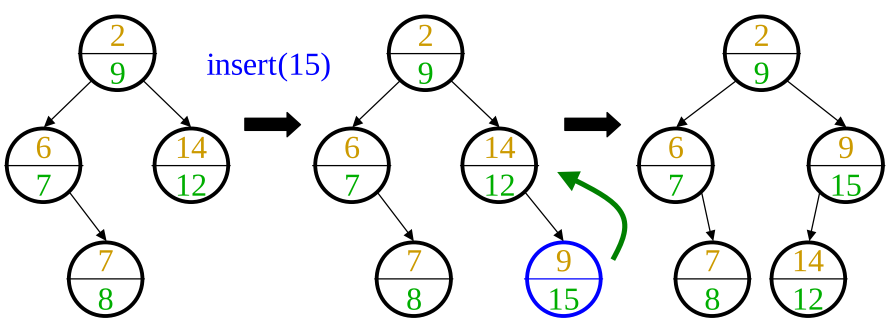
Heap in yellow; Search tree in green



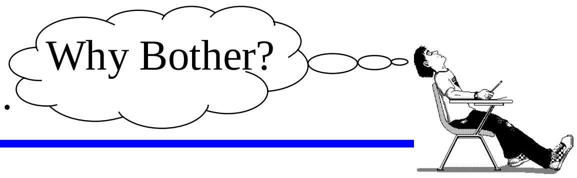


Treap Insert

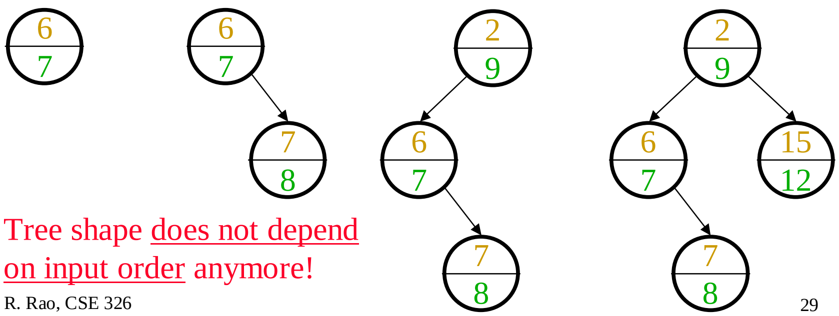
- Create node and assign it a random priority
- Insert as in normal BST
- Rotate up until heap order is restored (while maintaining BST property)



Tree + Heap…



- Inserting sorted data into a BST gives poor performance!
- Try inserting data in sorted order into a treap. What happens?



Tree shape <u>does not depend on input order</u> anymore!

Treap Summary

- Implements (randomized) Binary Search Tree ADT
  - Insert in expected O(log N) time for any input
  - Delete in expected O(log N) time for any input
    - Find the key and increase its value to ∞
    - Rotate it to the fringe
    - Snip it off
  - Find in expected O(log N) time for any input
  - But worst case is O(N)
- Memory use
  - O(1) per node
  - About the cost of AVL trees
- Very simple to implement, little overhead
  - Unlike AVL trees, no need to update balance information!

Final Example: Randomized Primality Testing

- Problem: Given a number N, is N prime?
  - Important for cryptography
- Randomized Algorithm based on a Result by Fermat:
  1. Guess a random number A, $0 < A < N$
  2. If (A^{N-1} mod N) ≠ 1, then Output "N is not prime"
  3. Otherwise, Output "N is (probably) prime"
     - N is prime with high probability but not 100%
     - N could be a "Carmichael number" – a slightly more complex test rules out this case (see text)
     - Can repeat steps 1-3 to make error probability close to 0
- Recent breakthrough: Polynomial time algorithm that is always correct (runs in $O(\log^{12} N)$ time for input N)
  - Agrawal, M., Kayal, N., and Saxena, N. "Primes is in P." Preprint, Aug. 6, 2002. http://www.cse.iitk.ac.in/primality.pdf

To Do:
Read Chapter 10 and
Sec. 12.5 (treaps)
Finish HW assignment #5

Next Time:
A Taste of Amortization
Final Review