CSE 484 (Winter 2008)

Applied Cryptography

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Goals for Today

- Asymmetric Cryptography
- CELT: Center for Engineering Learning and Teaching
- Reminder: Midterm on Friday. (Closed book.)
  - Contents up through the material for Monday (through symmetric crypto)
  - Not as hard as last year's midterm.
  - Make sure you understand the core concepts so far in this course:

Requirements for Public-Key Crypto

- Key generation: computationally easy to generate a pair (public key PK, private key SK)
  - Computationally infeasible to determine private key SK given only public key PK
- Encryption: given plaintext M and public key PK, easy to compute ciphertext $C=E_{PK}(M)$
- Decryption: given ciphertext $C=E_{PK}(M)$ and private key SK, easy to compute plaintext M
  - Infeasible to compute M from C without SK
  - Even infeasible to learn partial information about M
  - Trapdoor function: Decrypt(SK,Encrypt(PK,M))=M

Some Number Theory Facts ("Skip")

- Euler totient function $\varphi(n)$ where $n \geq 1$ is the number of integers in the [1,n] interval that are relatively prime to n
  - Two numbers are relatively prime if their greatest common divisor (gcd) is 1
- Euler's theorem:
  if $a \in Z_n^*$, then $a^{\varphi(n)}=1 \bmod n$
- Special case: Fermat's Little Theorem
  if p is prime and gcd(a,p)=1, then $a^{p-1}=1 \bmod p$

RSA Cryptosystem ("Fast")[Rivest, Shamir, Adleman 1977]

- Key generation:
  - Generate large primes p, q
    - Say, 1024 bits each (need primality testing, too)
  - Compute n=pq and $\varphi(n)=(p-1)(q-1)$
  - Choose small e, relatively prime to $\varphi(n)$
    - Typically, e=3 or $e=2^{16}+1=65537$ (why?)
  - Compute unique d such that ed = 1 mod $\varphi(n)$
  - Public key = (e,n); private key = d
- Encryption of m: $c = m^e \bmod n$
  - Modular exponentiation by repeated squaring
- Decryption of c: $c^d \bmod n = (m^e)^d \bmod n = m$

Why RSA Decryption Works ("Fast")

- $e \cdot d=1 \bmod \varphi(n)$
- Thus $e \cdot d=1+k \cdot \varphi(n)=1+k(p-1)(q-1)$ for some k

- Let m be any integer in Z_n
- If gcd(m,p)=1, then $m^{ed}=m \bmod p$
  - By Fermat's Little Theorem, $m^{p-1}=1 \bmod p$
  - Raise both sides to the power k(q-1) and multiply by m
  - $m^{1+k(p-1)(q-1)}=m \bmod p$, thus $m^{ed}=m \bmod p$
  - By the same argument, $m^{ed}=m \bmod q$
- Since p and q are distinct primes and $p \cdot q=n$, $m^{ed}=m \bmod n$

Why Is RSA Secure? ("Fast")

- RSA problem: given n=pq, e such that gcd(e,(p-1)(q-1))=1 and c, find m such that $m^e=c \bmod n$
  - i.e., recover m from ciphertext c and public key (n,e) by taking e^{th} root of c
  - There is no known efficient algorithm for doing this
- Factoring problem: given positive integer n, find primes $p_1, \ldots, p_k$ such that $n=p_1^{e_1}p_2^{e_2}\ldots p_k^{e_k}$
- If factoring is easy, then RSA problem is easy, but there is no known reduction from factoring to RSA
  - It may be possible to break RSA without factoring n

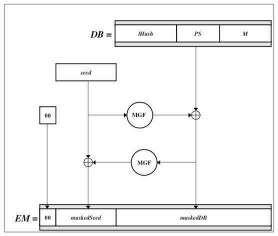
Caveats ("Fast;" Note first bullet)

- Don't use RSA directly
- e =3 is a common exponent
  - If $m < n^{1/3}$, then $c = m^3 < n$ and can just take the cube root of c to recover m
    - Even problems if "pad" m in some ways [Hastad]
  - Let $c_i = m^3 \bmod n_i$ - same message is encrypted to three people
    - Adversary can compute $m^3 \bmod n_1n_2n_3$ (using CRT)
    - Then take ordinary cube root to recover m

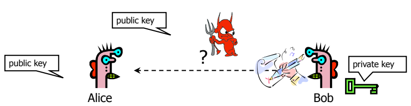
Integrity in RSA Encryption

- Plain RSA does <u>not</u> provide integrity
  - Given encryptions of m_1 and m_2, attacker can create encryption of $m_1 \cdot m_2$
    - $(m_1^e) \cdot (m_2^e) \bmod n = (m_1 \cdot m_2)^e \bmod n$
  - Attacker can convert m into m^k without decrypting
    - $(m_1^e)^k \bmod n = (m^k)^e \bmod n$
- In practice, OAEP is used: instead of encrypting M, encrypt $M \oplus G(r)$; $r \oplus H(M \oplus G(r))$
  - r is random and fresh, G and H are hash functions
  - Resulting encryption is plaintext-aware: infeasible to compute a valid encryption without knowing plaintext
    - … if hash functions are "good" and RSA problem is hard

OAEP (image from PKCS #1 v2.1)



Digital Signatures: Basic Idea



<u>Given</u>: Everybody knows Bob's public key
Only Bob knows the corresponding private key

<u>Goal</u>: Bob sends a "digitally signed" message

1. To compute a signature, must know the private key
2. To verify a signature, enough to know the public key

RSA Signatures

- Public key is (n,e), private key is d
- To sign message m: $s = m^d \bmod n$
  - Signing and decryption are the same operation in RSA
  - It's infeasible to compute s on m if you don't know d
- To verify signature s on message m:
  $s^e \bmod n = (m^d)^e \bmod n = m$
  - Just like encryption
  - Anyone who knows n and e (public key) can verify signatures produced with d (private key)
- In practice, also need padding & hashing (why?)

Encryption and Signatures

- Books often say: Encryption and decryption are inverses, so use decryption as signatures
- That's a common view
  - True for the RSA primitive
- But not the cryptographic view
  - To really use RSA, we need padding
  - Some encryption schemes don't have natural signature analogs and vice versa.

Advantages of Public-Key Crypto

- Confidentiality without shared secrets
  - Very useful in open environments
  - No "chicken-and-egg" key establishment problem
    - With symmetric crypto, two parties must share a secret before they can exchange secret messages
    - Caveats to come
- Authentication without shared secrets
  - Use digital signatures to prove the origin of messages
- Reduce protection of information to protection of authenticity of public keys
  - No need to keep public keys secret, but must be sure that Alice's public key is <u>really</u> her true public key

Disadvantages of Public-Key Crypto

- Calculations are 2-3 orders of magnitude slower
  - Modular exponentiation is an expensive computation
  - Typical usage: use public-key cryptography to establish a shared secret, then switch to symmetric crypto
    - We'll see this in IPSec and SSL
- Keys are longer
  - 1024 bits (RSA) rather than 128 bits (AES)
- Relies on unproven number-theoretic assumptions
  - What if factoring is easy?
    - Factoring is <u>believed</u> to be neither P, nor NP-complete
  - (Of course, symmetric crypto also rests on unproven assumptions)