Introduction to LS/LS+

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Cutting Planes Methods

Given a 0-1 integer program:

- Consider feasibility instead of optimization.

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Given a 0-1 integer program:

- Consider feasibility instead of optimization.
- Relax given integer linear program and consider polytope satisfying constraints.
- Transform through “cuts” to integral hull of valid solutions.

Cone Formulation of LS

A cone K is a subset of $\mathbb{R}^n$ where $x \in K$ if and only if $cx \in K$ for any constant $c \geq 0$.

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Definition

K is the cone of feasible points given by the linear program.

Goal:

Determine if the set of integer points in K is nonempty.

Cone Formulation of LS

Definition

$x \in N(K)$ if and only if there is a matrix Y such that:

- The first row of Y is x.
- The diagonal of Y is x (corresponds to $x_i^2 = x_i$).
- Y is symmetric ($Y_{ij} = Y_{ji}$ corresponds to $x_i x_j = x_j x_i$).
- All rows of Y are in K (corresponds to multiplying by x_i).
- All $Y_0 - Y_i$ are in K (corresponds to multiplying by $1 - x_i$).

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- All $Y_0 - Y_i$ are in K (corresponds to multiplying by $1 - x_i$).

Goal:

Show that $N(K)$ is stronger than K.

Validity of Cuts

Theorem

If K^0 is convex hull of 0-1 vectors in K, then $K^0 \subseteq N(K) \subseteq K$.

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Proof

Trivially, $N(K) \subseteq K$.
If x is a 0-1 vector in K^0, the matrix $Y = xx^T$ satisfies all necessary constraints.

Proof

$$Y = xx^T =$$

$$\begin{pmatrix} x_0x_0 & \cdots & x_0x_i & \cdots & x_0x_n \\ \vdots & \ddots & & & \vdots \\ x_ix_0 & & x_ix_i & & x_ix_n \\ \vdots & & & \ddots & \vdots \\ x_nx_0 & \cdots & x_nx_i & \cdots & x_nx_n \end{pmatrix}$$

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- The diagonal is x ($x_i^2 = x_i$)
- Symmetric ($x_ix_j = x_jx_i$)
- Each row is in K ($x_i \cdot x$)
- $Y_0 - Y_i$ is in K ($(1 - x_i) \cdot x$).

Strength of Cuts

Theorem

n rounds of LS give the convex hull of 0-1 solutions (i.e. $K^0 = N^n(K)$).

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Proof ($K^0 \subseteq N^n(K)$)

From before, $K^0 \subseteq N(K) \subseteq K$.

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n rounds of LS give the convex hull of 0-1 solutions (i.e. $K^0 = N^n(K)$).

Proof ($K^0 \subseteq N^n(K)$)

From before, $K^0 \subseteq N(K) \subseteq K$.
Hence, $K^0 \subseteq N^k(K)$ for all k.

Strength of Cuts

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Theorem

- $v \in H_i$ if and only if $v_i = 0$
- $v \in G_i$ if and only if $v_i = v_0$
- $F_i = H_i \cup G_i$

Then, $N(K) \subseteq \text{cone}(K \cap F_i)$.

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Then, $N(K) \subseteq \text{cone}(K \cap F_i)$.

Proof

Let $x \in N(K)$, with corresponding Y.
Let Y_i be the ith row of Y.

Strength of Cuts

Proof (cont.)

$$\left(\begin{array}{cc|c|cc} Y_{00} & \cdots & Y_{0i} & \cdots & Y_{0n} \\ \vdots & \ddots & & & \vdots \\ \hline Y_{i0} & & Y_{ii} & & Y_{in} \\ \hline \vdots & & & \ddots & \vdots \\ Y_{n0} & \cdots & Y_{ni} & \cdots & Y_{nn} \end{array}\right)$$

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- $Y_{ii} = Y_{i0}$ implies $Y_i \in G_i$.

Strength of Cuts

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- $Y_i \in K \cap G_i$.

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Proof (cont.)

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- $Y_i \in K \cap G_i$.
- $(Y_0 - Y_i)_i = Y_{0i} - Y_{ii} = 0$ implies $(Y_0 - Y_i) \in H_i$.

Strength of Cuts

Proof (cont.)

$$\left(\begin{array}{cc|c|cc} Y_{00} & \cdots & Y_{0i} & \cdots & Y_{0n} \\ \vdots & \ddots & & & \vdots \\ \hline Y_{i0} & & Y_{ii} & & Y_{in} \\ \hline \vdots & & & \ddots & \vdots \\ Y_{n0} & \cdots & Y_{ni} & \cdots & Y_{nn} \end{array}\right)$$

- $Y_i \in K \cap G_i$.
- $(Y_0 - Y_i) \in K \cap H_i$,

Strength of Cuts

Proof (cont.)

So:

$$x = Y_0$$

Strength of Cuts

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So:

$$\begin{aligned} x &= Y_0 \\ &= (Y_0 - Y_i) + Y_i \end{aligned}$$

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$$\begin{aligned} x &= Y_0 \\ &= (Y_0 - Y_i) + Y_i \\ &\in (K \cap H_i) + (K \cap G_i) \end{aligned}$$

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Proof (cont.)

So:

$$\begin{aligned} x &= Y_0 \\ &= (Y_0 - Y_i) + Y_i \\ &\in (K \cap H_i) + (K \cap G_i) \\ &\subseteq \text{cone}(K \cap (H_i \cup G_i)) \end{aligned}$$

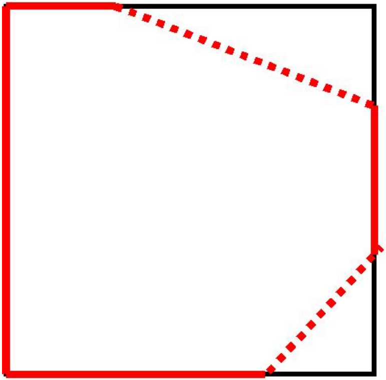
Strength of Cuts

Proof (cont.)

So:

$$
\begin{aligned}
x &= Y_0 \\
&= (Y_0 - Y_i) + Y_i \\
&\in (K \cap H_i) + (K \cap G_i) \\
&\subseteq \text{cone}(K \cap (H_i \cup G_i)) \\
&= \text{cone}(K \cap F_i)
\end{aligned}
$$

Example



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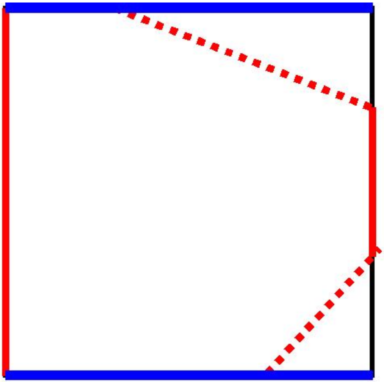

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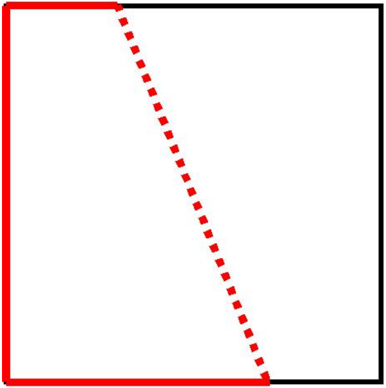

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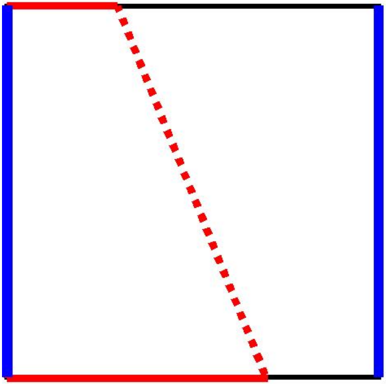

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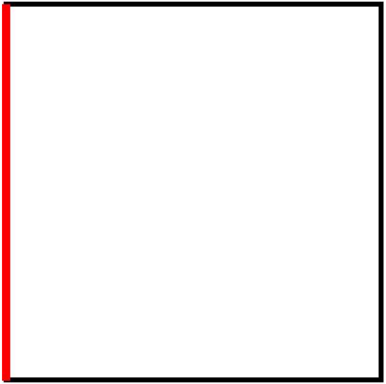

Conclusion

In terms of cones:

$$N^t(K) = N(N^{t-1}(K)) \subseteq \text{cone}(N^{t-1}(K) \cap F_t)$$

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Since $\cap_1^n F_t$ is the vertices of hypercube,
$N^n(K) \subseteq K^0$.

LS+

LS+ strengthens an LP in the same way as LS,
also adds squares of linear terms.

Cone Formulation of LS+

Let $x \in N_+(K)$ if and only if there is a matrix Y such that:

- The first column of Y is x
- The diagonal of Y is x
- Y is symmetric $Y_{ij} = Y_{ji}$
- All the rows of Y are in K
- For all i, $Y_0 - Y_i$ is in K
- Y is positive semidefinite.

Cone Formulation of LS+

Recall: Y positive semidefinite means: $v^T Y v \geq 0$ for all v.

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Let A, B be positive semidefinite matrices.

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Let A, B be positive semidefinite matrices.

- $v^T(cA)v = c(v^T A v) \geq 0$, so cA is positive semidefinite.
- $v(A+B)v = v^T A v + v^T B v \geq 0$, so $A+B$ is positive semidefinite.

Cone Formulation of LS+

Recall: Y positive semidefinite means: $v^T Yv \geq 0$ for all v.
Let A, B be positive semidefinite matrices.

- $v^T(cA)v = c(v^T Av) \geq 0$, so cA is positive semidefinite.
- $v(A + B)v = v^T Av + v^T Bv \geq 0$, so $A + B$ is positive semidefinite.

$N_+(K)$ is a convex cone.

SDP

A semi-definite program is of the form:

$$
\begin{aligned}
\min C \circ Y \quad & \text{s.t.} \\
A_1 \circ Y \quad &= \quad b_1 \\
&\vdots \\
A_m \circ Y \quad &= \quad b_m \\
Y \succeq 0 &
\end{aligned}
$$

Where $C, A_1, \ldots, A_m$ are symmetric matrices, and $b_1, \ldots, b_m$ are scalars.

Vertex Cover

LP:

$$\min \sum_{i \in V} x_i$$

$$\begin{aligned} x_i + x_j - 1 \geq 0 &\qquad \text{for all } (i,j) \in E \\ 0 \leq x_i \leq 1 &\qquad \text{for all } i. \end{aligned}$$

Vertex Cover

By applying LS lift rules:

$$(1 - x_i)(1 - x_j) \geq 0 \text{ for all } i, j.$$

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Since $x_i^2 = x_i$

$$\begin{aligned} 0 &\leq (1 - x_i)(x_i + x_j - 1) \\ &= (1 - x_i)(x_j - 1) \text{ for all } (i, j) \in E. \end{aligned}$$

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Or equivalently

$$(1 - x_i)(1 - x_j) \leq 0 \text{ for all } (i, j) \in E.$$

Vertex Cover

Let $x_0 = 1$.

$$\min \sum_{i \in V} (x_0 x_i)$$

$$(x_0 - x_i)(x_0 - x_j) = 0 \qquad \text{for all } (i,j) \in E$$
$$(x_0 - x_i)(x_0 - x_j) \geq 0 \qquad \text{for all } i, j.$$

Vertex Cover

Let $Y = U^T U$ be LS+ lifted matrix.
Let the columns of Y be $(u_0, \dots, u_n)$.

Vertex Cover

Let $Y = U^T U$ be LS+ lifted matrix.
Let the columns of Y be $(u_0, \ldots, u_n)$. Then:

$$\min \sum_{i \in V} u_0 \cdot u_i$$

$$(u_0 - u_i) \cdot (u_0 - u_j) = 0 \qquad \text{for all } (i,j) \in E$$
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Let $v_i = 2u_i - u_0$.

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$$\begin{aligned} ||v_i||^2 &= v_i \cdot v_i \\ &= 4u_i \cdot u_i - 4u_i \cdot u_0 + u_0 \cdot u_0 \\ &= ||u_0||^2 = 1 \end{aligned}$$

Vertex Cover

SDP

$$\min \sum_{i \in V} \frac{1 + v_0 \cdot v_i}{2}$$

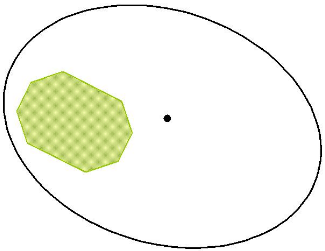
$$\begin{aligned} (v_0 - v_i) \cdot (v_0 - v_j) &= 0 \quad \text{for all} \quad ij \in E \\ (v_0 - v_i) \cdot (v_0 - v_j) &\geq 0 \quad \text{for all} \quad i \in V \\ ||v_i|| &= 1 \quad \text{for all} \quad i \in V. \end{aligned}$$

Separation Oracles

Definition

strong separation oracle - given a point x, returns that $x \in K$, or gives a separating hyperplane.

Ellipsoid Method for $N(K)$



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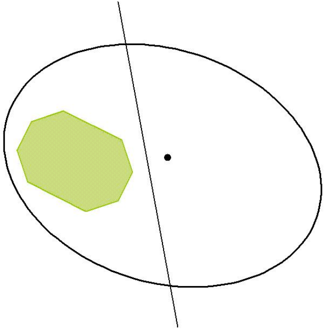

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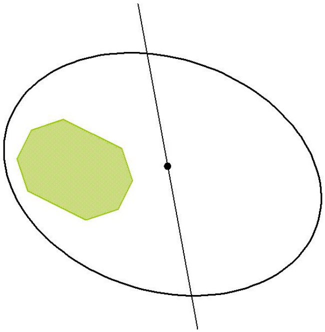

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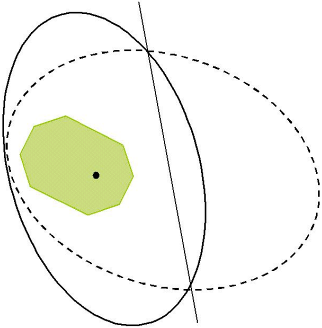

Volume of ellipse decreases by $2^{\frac{1}{2n+2}}$.

Ellipsoid Method for $N_+(K)$?

$N_+(K)$ not necessarily polyhedral.

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Definition

weak separation oracle - given a point x and $\epsilon > 0$, returns that $dist(x, K) < \epsilon$ or returns a separating hyperplane h such that $dist(h, K^{\perp}) < \epsilon$.

Separation Problem

Theorem

Given a weak separation oracle for K, we can solve the weak separation problem for $N_+(K)$ in polynomial time.

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First row, diagonal, and symmetry conditions can be checked trivially.

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First row, diagonal, and symmetry conditions can be checked trivially.
Positive semi-definitiveness checked by Gaussian elimination.

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Proof

Can solve for space of Y matrices.
First row, diagonal, and symmetry conditions can be checked trivially.
Positive semi-definitiveness checked by Gaussian elimination.
Both row conditions given by separation oracle.

Separation Problem