CSE 326 Data Structures

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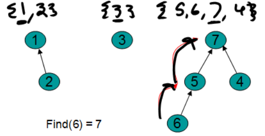
Disjoint Sets

Logistics

- Homework 5 on web due Friday
- Project 3 out, "Shake-n-Bacon" code due Mon, Feb 26
- Reading: finish Weiss Chapter 8, start Chapter 7 Sorting
- Lecturer Friday: * Ruth Anderson
- Holiday Monday!

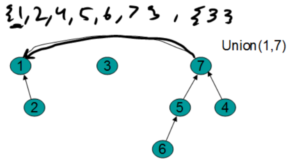
Find Operation

Find(x) - follow x to the root and return the root

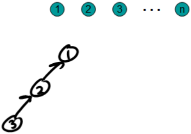


Union Operation

Union(x,y) - assuming x and y are roots, point y to x.



A Bad Case



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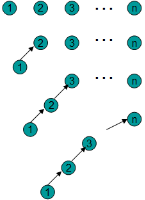


Union(2,1)

Union(3,2)

:

Union(n,n-1)

Find(1) n steps!!

Now this doesn't look good ☹

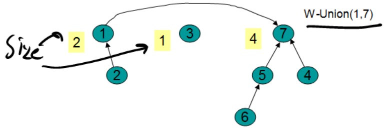
Can we do better? *Yes!* m finds
n-1 unions

1. Improve union so that *find* only takes $O(\log n)$
   - Union-by-size
   - Reduces complexity to $O(m \log n + n)$

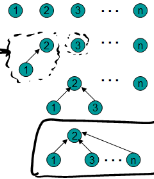
2. Improve find so that it becomes even better!
   - Path compression
   - Reduces complexity to "almost" $O(m + n)$

Weighted Union

- Weighted Union
  - Always point the *smaller* (total # of nodes) tree to the root of the larger tree



Example Again



W-Union(2,1)

W-Union(3,2)

⋮

W-Union(n,2)

smaller

↓

larger

height

Find(1) constant time

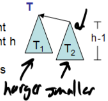
Find (1)

Analysis of Weighted Union

With weighted union an up-tree of height h has weight *at least* 2^h.

- Proof by induction 0
  - **Basis**: h = 0. The up-tree has one node, $2^0 = 1$
  - **Inductive step**: Assume true for all h' < h.

Minimum weight up-tree of height h formed by weighted unions



$W(T_1) \geq W(T_2) \geq 2^{h-1}$

Weighted union

Induction hypothesis

$W(\mathbf{T}) \geq 2^{h-1} + 2^{h-1} = 2^h$

$W(T) = W(T_1) + W(T_2)$

Analysis of Weighted Union (cont)

Let T be an up-tree of weight n formed by weighted union. Let h be its height.

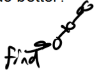
n elements

$n \geq 2^h$

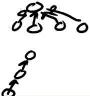
$\log_2 n \geq h$

$h \leq \log_2 n$

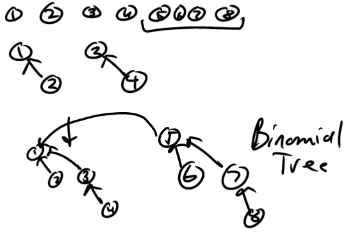
- Find(x) in tree T takes O(log n) time.
  - Can we do better?



height

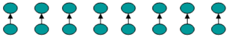


Worst Case for Weighted Union

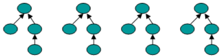


Worst Case for Weighted Union

n/2 Weighted Unions

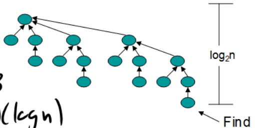


n/4 Weighted Unions



Example of Worst Cast (cont')

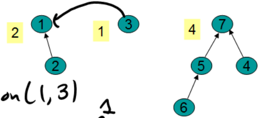
After n/2 + n/4 + …+ 1 Weighted Unions:



n elements
height O(logn)

If there are $n = 2^k$ nodes then the longest path from leaf to root has length k.

Array Implementation



Union(1,3)

	1	2	3	4	5	6	7
up	-1	1	~~-1~~ 1	7	7	5	-1
weight	~~2~~ 3		~~1~~ 0				4

Weighted Union

```
W-Union(i,j : index){
  //i and j are roots
  wi := weight[i];
  wj := weight[j];
  if wi < wj then
    up[i] := j;
    weight[j] := wi + wj;
  else
    up[j] :=i;
    weight[i] := wi +wj;
}
```

new runtime for Union():

$O(1)$

new runtime for Find():

$O(\log n)$

runtime for m finds and n-1 unions =

$O(m \log n + n)$

Nifty Storage Trick

- Use the same array representation as before
- Instead of storing `-1` for the root, simply store `-size`

[Read section 8.4, page 276]

How about Union-by-height?

- Can still guarantee O(log *n*) worst case depth

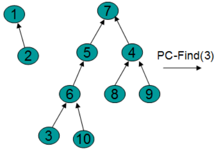
  Left as an exercise!

- Problem: Union-by-height doesn't combine very well with the new find optimization technique we'll see next

"Path Compression"

Path Compression

- On a Find operation point all the nodes on the search path directly to the root.



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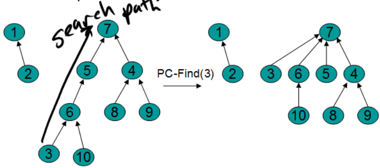
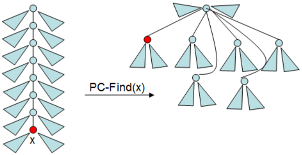


Self-Adjustment Works

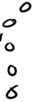


Path Compression Find

```
PC-Find(i : index) {
  r := i;
  while up[r] ≠ -1 do //find root//
    r := up[r];
  if i ≠ r then  //compress path//
    k := up[i];
    while k ≠ r do
      up[i] := r;
      i := k;
      k := up[k]
  return(r)
}
```

find root

compress



Interlude: A Really Slow Function

$n = 2^h$

Ackermann's function is a <u>really</u> big function $A(x, y)$ with inverse $\alpha(x, y)$ which is <u>really</u> small

How fast does $\alpha(x, y)$ grow?

$\alpha(x, y) = 4$ for x **far** larger than the number of atoms in the universe (2^{300})

α shows up in:

- Computation Geometry (surface complexity)
- Combinatorics of sequences

A More Comprehensible Slow Function

$\log^* x$ = number of times you need to compute log to bring value down to at most 1

$\log^* 4 = 2$

$\log \log 4 = \log 2 = 1$

E.g. $\log^* 2 = 1$

$\log^* 4 = \log^* 2^2 = 2$

$\log^* \underline{16} = \log^* 2^{2^2} = 3$ (log log log 16 = 1)

$\log^* \underline{65536} = \log^* 2^{2^{2^2}} = 4$ (log log log log 65536 = 1)

$\log^* 2^{65536} = \ldots\ldots\ldots\ldots\ldots = 5$

1 2

2 2^2

3 2^{2^2}

4 $2^{2^{2^2}}$

Take this: $\alpha(m,n)$ grows even slower than $\log^* n$!!

Complex Complexity of Union-by-Size + Path Compression

Tarjan proved that, with these optimizations, p union and find operations on a set of n elements have worst case complexity of $O(p \cdot \alpha(p, n))$

$\alpha = 4$

$O(p \cdot$ so slow$)$

For *all practical purposes* this is amortized constant time:

$O(p \cdot 4)$ for p operations!

$$\frac{O(p \cdot 4)}{p} = O(1)$$

- Very complex analysis – worse than splay tree analysis etc. that we skipped!

Disjoint Union / Find with Weighted Union and PC

n unions
m finds

- Worst case time complexity for a W-Union is O(1) and for a PC-Find is O(log n).
- Time complexity for $m \geq n$ operations on n elements is O(m log* n) where log* n is a very slow growing function.
  - Log * n < 7 for all reasonable n. Essentially constant time per operation!
- Using "ranked union" gives an even better bound theoretically.

Amortized Complexity

- For disjoint union / find with weighted union and path compression.
  - average time per operation is essentially a constant.
  - worst case time for a PC-Find is O(log n).
- An individual operation can be costly, but over time the average cost per operation is not.