Number Systems

- Readings: 3-3.3.3, 3.3.5
- Problem: Implement simple pocket calculator
- Need: Display, adders & subtractors, inputs
  - Display: Seven segment displays
  - Inputs: Switches
- Missing: Way to implement numbers in binary

- Approach: From decimal to binary numbers (and back)

Arithmetic Operations

Decimal:

```
  5 7 8 9 2
+ 7 8 9 5 6
```

Binary:

```
    1 0 1 0 1 1 1
+ 0 1 0 0 1 0 1
```

Decimal:

```
  5 7 8 9 2
- 3 2 9 4 6
```

Binary:

```
    1 0 1 0 0 1 1 0
- 0 0 1 1 0 1 1 1
```

Arithmetic Operations (cont.)

Decimal:

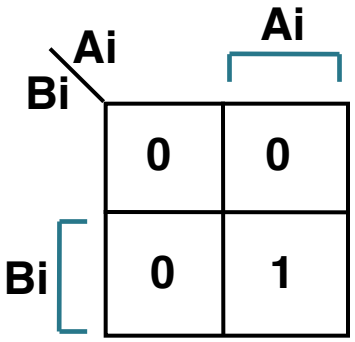
```
  2 0 1
* 2 1 4
```

Binary:

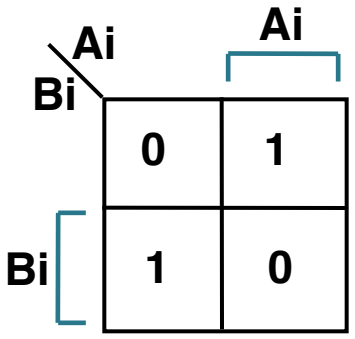
```
  1 0 0 1
* 1 0 1 1
```

Half Adder

Ai	Bi	Carry	Sum
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

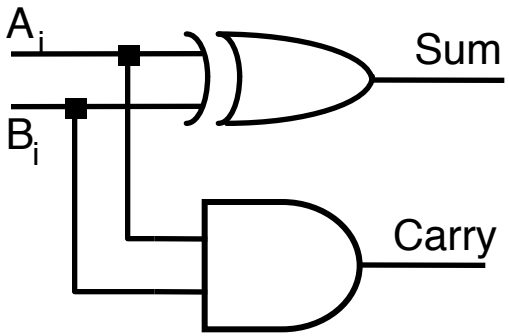


Carry = Ai Bi



Sum = $\overline{Ai}$ Bi + Ai $\overline{Bi}$

= Ai ⊕ Bi



Half-adder Schematic

Full Adder

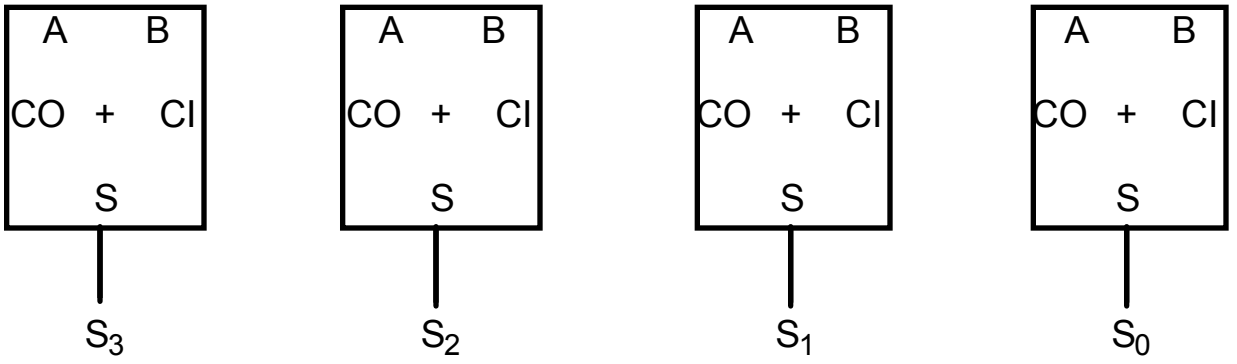
A	B	CI	CO	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Full Adder Implementation

Multi-Bit Addition

A_3 B_3 A_2 B_2 A_1 B_1 A_0 B_0

$$\begin{array}{r} A_3\ A_2\ A_1\ A_0 \\ +\ B_3\ B_2\ B_1\ B_0 \\ \hline \end{array}$$



Multi-Bit Addition in Verilog, Parameters

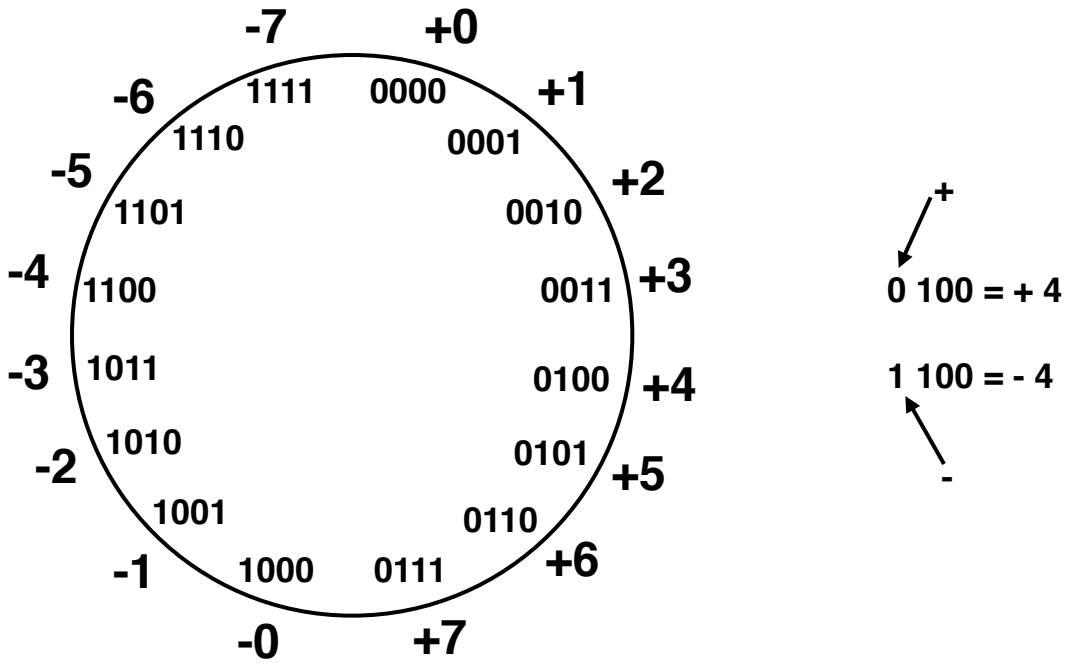
```
module uadd #(parameter WIDTH=8)
   (out, a, b);
   output reg [WIDTH:0] out;
   input      [WIDTH-1:0] a, b;

   always @(*) begin
     out = a + b;
   end
endmodule

module add4 #(parameter W=22)
   (out, a, b, c, d);
   output [W+1:0] out;
   input  [W-1:0] a, b, c, d;
```

```
endmodule
```

Negative Numbers

- Need an efficient way to represent negative numbers in binary
  - Both positive & negative numbers will be strings of bits
  - Use fixed-width formats (4-bit, 16-bit, etc.)
- Must provide efficient mathematical operations
  - Addition & subtraction with potentially mixed signs
  - Negation (multiply by -1)

Sign/Magnitude Representation



High order bit is sign: 0 = positive (or zero), 1 = negative

Three low order bits is the magnitude: 0 (000) thru 7 (111)

Number range for n bits = $+/-2^{n-1} - 1$

Representations for 0:

Sign/Magnitude Addition

```
   0 0 1 0 (+2)
+  0 1 0 0 (+4)
```

```
   1 0 1 0 (-2)
+  1 1 0 0 (-4)
```

```
   0 0 1 0 (+2)
+  1 1 0 0 ( -4)
```

```
   1 0 1 0 ( -2)
+  0 1 0 0 (+4)
```

Bottom line: Basic mathematics are too complex in Sign/Magnitude

Idea: Pick negatives so that addition works

- Let -1 = 0 - (+1):

```
  0  0  0  0  (  0)
- 0  0  0  1  (+1)
```

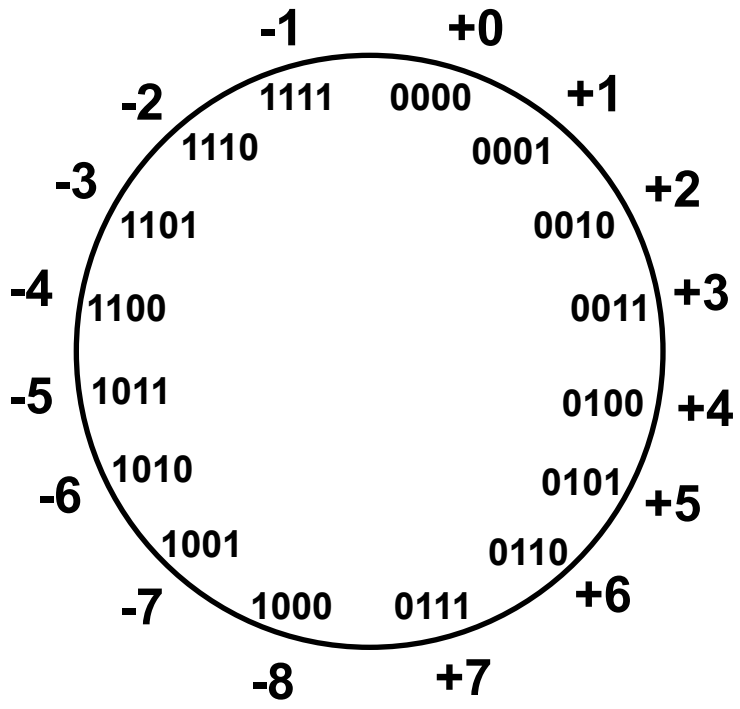
- Does addition work?

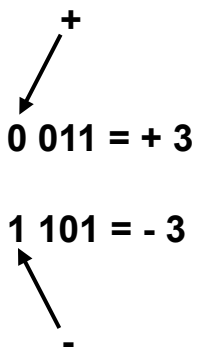
```
  0  0  1  0  (+2)
+ 1  1  1  1  ( -1)
```

- Result: Two's Complement Numbers

Two's Complement

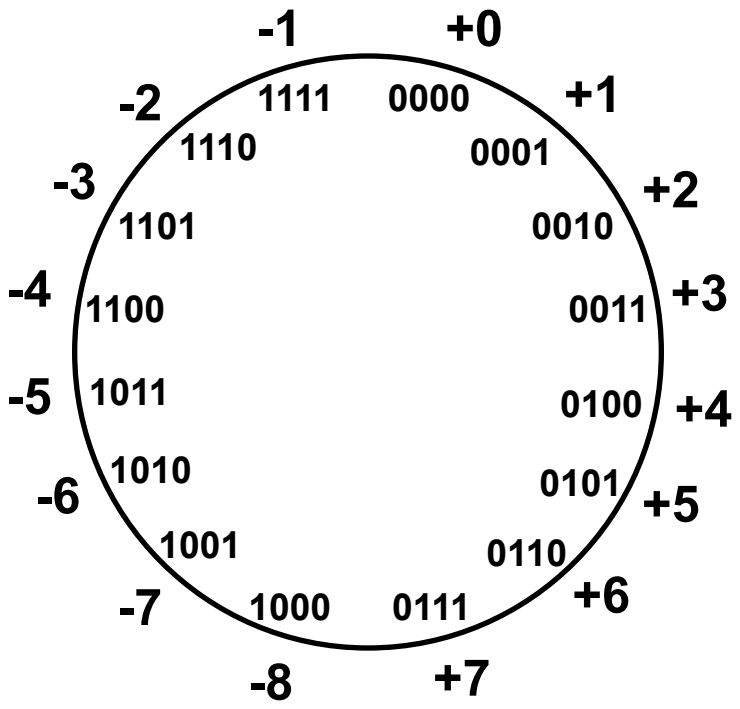
- Only one representation for 0
- One more negative number than positive number
- Fixed width format for both pos. & neg. numbers





Negating in Two's Complement

- Flip bits & Add 1
- Negate $(0010)_2$ (+2)



- Negate $(1110)_2$ (-2)

Addition in Two's Complement

```
   0 0 1 0 (+2)
+  0 1 0 0 (+4)
```

```
   1 1 1 0 (-2)
+  1 1 0 0 (-4)
```

```
   0 0 1 0 (+2)
+  1 1 0 0 ( -4)
```

```
   1 1 1 0 ( -2)
+  0 1 0 0 (+4)
```

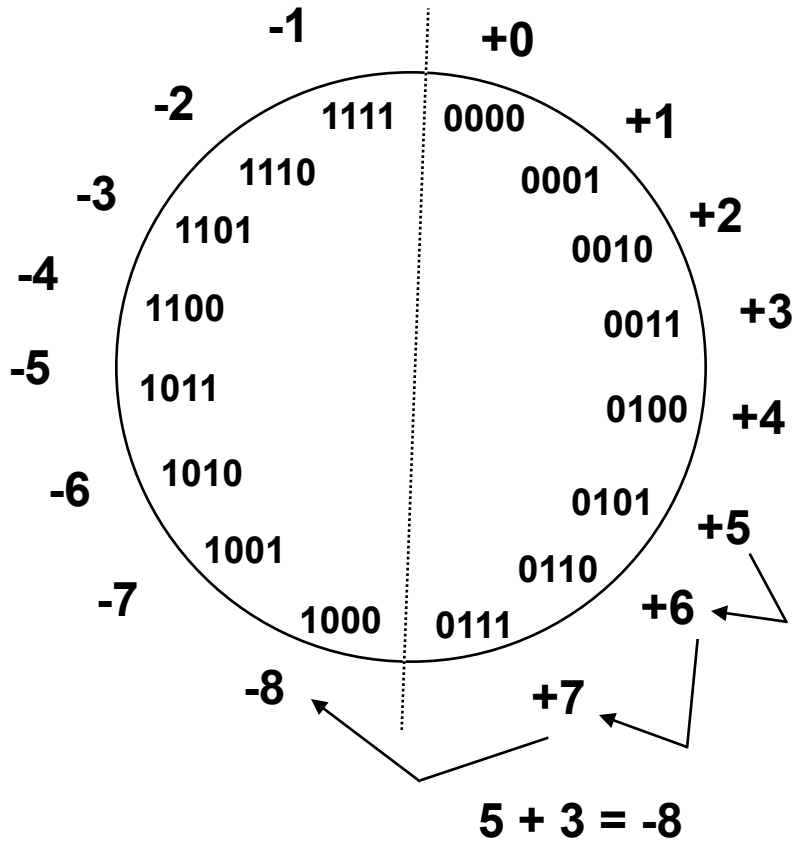
Subtraction in Two's Complement

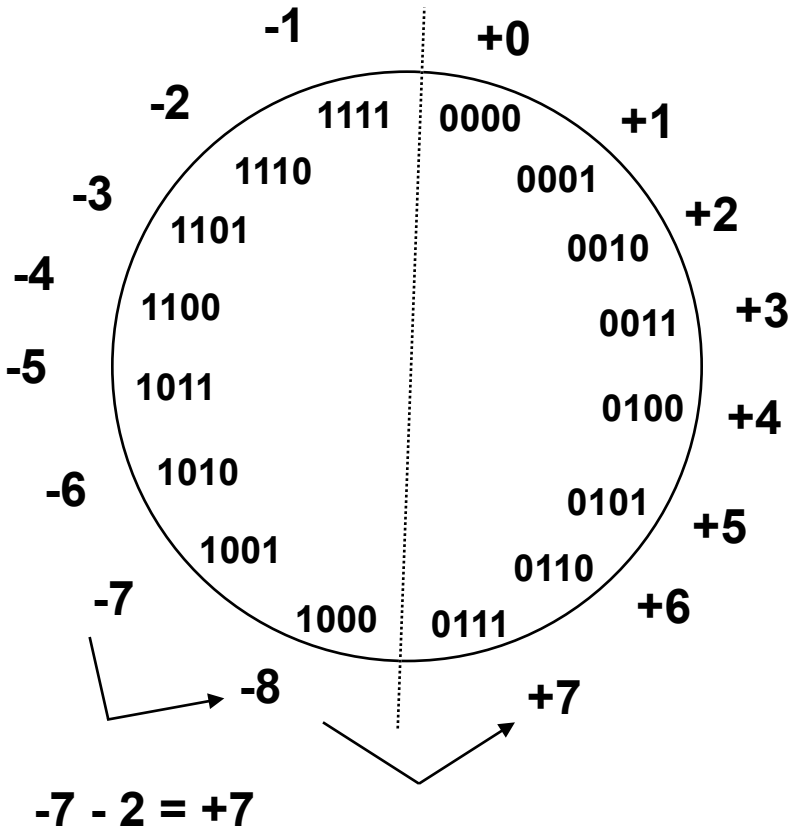
- $A - B = A + (-B) = A + \overline{B} + 1$
- 0010 - 0110
- 1011 - 1001
- 1011 - 0001

Overflows in Two's Complement

Add two positive numbers but get a negative number

or two negative numbers but get a positive number





Overflow Detection in Two's Complement

5	0 1 0 1
3	0 0 1 1
-8	

Overflow

-7	1 0 0 1
-2	1 1 1 0
7	

Overflow

5	0 1 0 1
2	0 0 1 0
7	

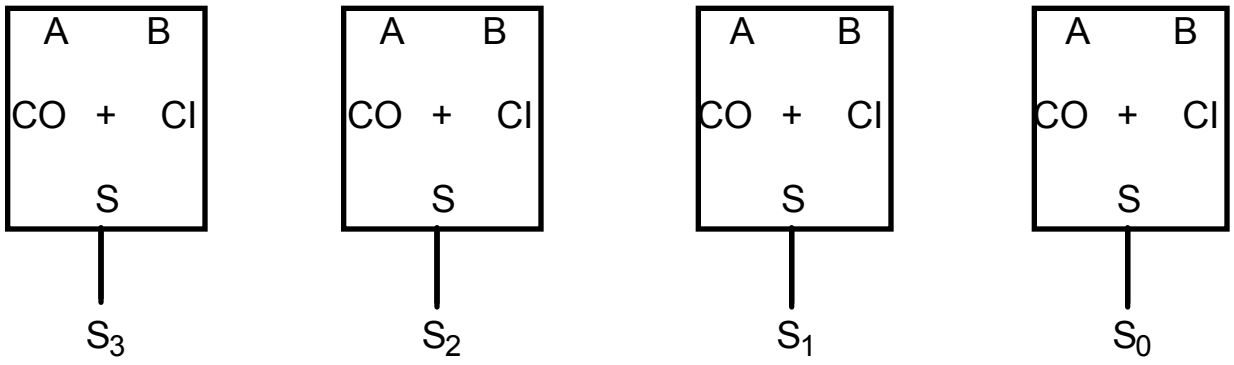
No overflow

-3	1 1 0 1
-5	1 0 1 1
-8	

No overflow

Adder/Subtractor

A_3 B_3 A_2 B_2 A_1 B_1 A_0 B_0



Overflow

$$A - B = A + (-B) = A + \overline{B} + 1$$

Multi-Bit Addition in Verilog, Parameters

```
module uadd #(parameter WIDTH=8)
  (out, a, b);
  output reg [WIDTH:0] out;
  input      [WIDTH-1:0] a, b;

  always @(*) begin
    out = a + b;
  end
endmodule

module add4 #(parameter W=22)
  (out, a, b, c, d);
  output [W+1:0] out;
  input  [W-1:0] a, b, c, d;

  wire [W:0] ab, cd;

  uadd #(.WIDTH(W))   u_ab   (ab, a, b);
  uadd #(.WIDTH(W))   u_cd   (cd, c, d);
  uadd #(.WIDTH(W+1)) u_abcd (out, ab, cd);
endmodule
```

Converting Decimal to Two's Complement

- Convert absolute value to unsigned binary, then fixed width, then negate if necessary

- Convert $(-9)_{10}$ to 6-bit Two's Complement

- Convert $(9)_{10}$ to 6-bit Two's Complement

Converting Two's Complement to Decimal

- If Positive, convert as normal;
  If Negative, negate then convert.

- Convert $(11010)_2$ to Decimal

- Convert $(01101)_2$ to Decimal

Sign Extension

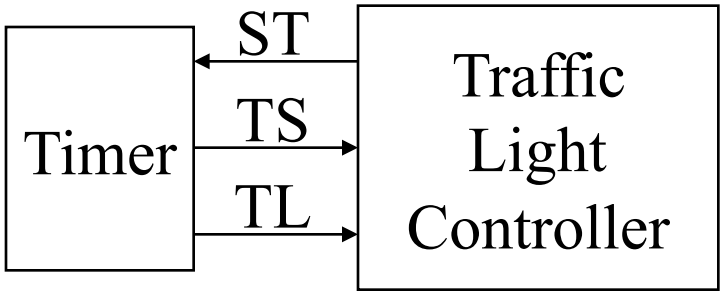
- To convert from N-bit to M-bit Two's Complement (N<M), simply duplicate sign bit:

- Convert $(0010)_2$ to 8-bit Two's Complement

- Convert $(1011)_2$ to 8-bit Two's Complement

Solving Complex Problems

- Many problems too complex to build as one system
  - Replace with communicating sub-circuits



- Design process:
  - Understand the problem
  - Break problem into subsystems, identifying connections
  - Design individual subsystems.

Complex Problem Example

- Design a digital clock, which can
  - Display the seconds, minutes and hours
  - Have three inputs
    - Increment hour
    - Increment minute
    - Reset seconds

Complex Problem Example (cont.)

Complex Problem Example (cont.)

Complex Problem Example (cont.)

Complex Problem Example

- Break into pieces:
  - Display, counter
  - Displayer becomes 6x 7-segment displays
  - Counter becomes three counters
    - Minutes, hours, seconds.
  - Need reset on seconds, override on increment on hours, minutes.
  - Break counters into digits, except hours.
  - Communicate increment to higher