Lattice-Theoretic Data Flow Analysis Framework

Goals:

- provide a single, formal model that describes all DFAs
- formalize notions of “safe”, “conservative”, “optimistic”
- place precise bounds on time complexity of DF analysis
- enable connecting analysis to underlying semantics for correctness proofs

Plan:

- define **domain** of program properties computed by DFA
  - domain: set of elements + order over elements = **lattice**
- define flow functions & merge function over this domain, using standard lattice operators
- benefit from lattice theory in attacking above issues

History: Kildall [POPL 73], Kam & Ullman [JACM 76]

Lattices

Define lattice *D* = (*S*, ≤):

- *S* is a (possibly infinite) set of elements
- ≤ is a binary relation over elements of *S*

Required properties of ≤:

- ≤ is a **partial order**
  - reflexive, transitive, & anti-symmetric
- every pair of elements of *S* has
  a unique **greatest lower bound** (a.k.a. meet) and
  a unique **least upper bound** (a.k.a. join)

Height of *D* =
longest path through partial order from greatest to least

- convenient to count edges, not nodes
- infinite lattice can have finite height (but infinite width)

Top (T) = unique element of *S* that’s greatest, if exists
Bottom (⊥) = unique element of *S* that’s least, if exists

Lattice models in data flow analysis

Model data flow information by an element of a lattice domain

- our convention: if *a* < *b*, then *a* is **less precise** than *b*
  - i.e., *a* is a conservative approximation to *b*
- top = most precise, best case info
- bottom = least precise, worst case info
- merge function = g.l.b. (meet) on lattice elements (the most precise element that’s a conservative approximation to both input elements)
- initial info for optimistic analysis (at least back edges): top

(Reverse less precise/more precise conventions used in PL semantics, abstract interpretation!)

Examples

Reaching definitions:

- an element:
- set of all elements:
- ≤:
- top:
- bottom:
- meet:

Reaching constants:

- an element:
- set of all elements:
- ≤:
- top:
- bottom:
- meet:

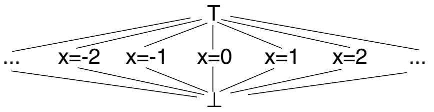
Some typical lattice domains

Powerset lattice: set of all subsets of a set *S*

- ordered by $\subseteq$ or $\supseteq$
- top & bottom = $\varnothing$ & *S*, or vice versa
- height = $|S|$ (infinite if *S* is infinite)
- “a collecting analysis”

A lifted set: a set of incomparable values, plus top & bottom

- e.g., reaching constants domain, for a particular variable:



- height = 2 [edges] (even though width is infinite!)

Two-point lattice: top and bottom

- computes a boolean property

Single-point lattice: just bottom

- trivial do-nothing analysis

Tuples of lattices

Often helpful to break down a complex lattice into a tuple of lattices, one per variable/stmt/... being analyzed

Formally: $D_T = <S_T, \leq_T> \ = \ (D = <S, \leq>)^N$

- $S_T = S_1 \times S_2 \times ... \times S_N$
  - element of tuple domain is a tuple of elements from each variable’s domain
  - i^{th} component of tuple is info about i^{th} variable/stmt/...
- $<..., d_{1i}, ...> \ \leq_T \ <..., d_{2i}, ...> \ \equiv \ d_{1i} \leq d_{2i}, \forall i$
  - i.e. **pointwise** ordering
- meet: pointwise meet
- top: tuple of tops
- bottom: tuple of bottoms
- height(D_T) = *N* * height(D)

Powerset(*S*) lattice is isomorphic to a tuple of two-point lattices, one two-point lattice element per element of *S*

- i.e., a bit-vector!

Example: reaching constants

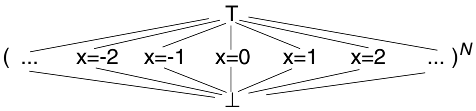
How to model reaching constants for all variables?

Informally:
each element is a set of the form {..., $x \to k$, ...}, with at most one binding for x

One lattice model: a powerset of all $x \to k$ bindings

- S = pow({ $x \to k$ | $\forall x$, $\forall k$ })
- $\leq \, = \, \subseteq$
- height?

Another lattice model:
N-tuple of 3-level constant prop. lattices, for each of *N* variables

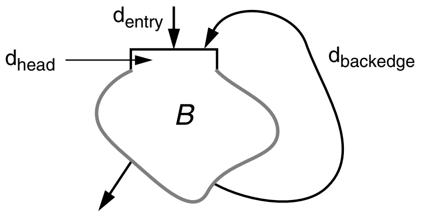
- 

- height?

If not, which is better?

Analysis of loops in lattice model

Consider:



(Assume $B(d_{head})$ computes $d_{backedge}$)

Want solution to constraints:

$d_{head} = d_{entry} \cap d_{backedge}$ [$\cap$ means meet]
$d_{backedge} = B(d_{head})$

Let $F(d) = d_{entry} \cap B(d)$

Then want fixed-point of *F*:

$d_{head} = F(d_{head})$

Iterative analysis in lattice model

Iterative analysis computes fixed-point
by iterative approximation, beginning with T:

$F^0 = \mathrm{d}_{\mathrm{entry}} \cap \mathrm{T} = \mathrm{d}_{\mathrm{entry}}$

$F^1 = \mathrm{d}_{\mathrm{entry}} \cap B(F^0) = F(F^0) = F(\mathrm{d}_{\mathrm{entry}})$

$F^2 = \mathrm{d}_{\mathrm{entry}} \cap B(F^1) = F(F^1) = F(F(F^0)) = F(F(\mathrm{d}_{\mathrm{entry}}))$

. . .

$F^k = \mathrm{d}_{\mathrm{entry}} \cap B(F^{k-1}) = F(F^{k-1}) = F(F(...(F(\mathrm{d}_{\mathrm{entry}}))...))$

until

$F^{k+1} = \mathrm{d}_{\mathrm{entry}} \cap B(F^k) = F(F^k) \quad = \quad F^k$

Is *k* finite?
If so, how big can it be?

Termination of iterative analysis

In general, *k* need not be finite

Sufficient conditions for finiteness:

- flow functions (e.g. *F*) are **monotonic**
- lattice is of finite **height**

A function *F* is monotonic iff:
$\mathrm{d}_2 \le \mathrm{d}_1 \ \Rightarrow\ F(\mathrm{d}_2) \le F(\mathrm{d}_1)$

- for application of DFA, this means that giving a flow function at least as conservative inputs ($\mathrm{d}_2 \le \mathrm{d}_1$) leads to at least as conservative outputs ($F(\mathrm{d}_2) \le F(\mathrm{d}_1)$)

For monotonic *F* over domain *D*, the maximum number of times that *F* can be applied to itself, starting w/ any element of D, w/o reaching fixed-point, is height(*D*)

- start at top of *D*
- for each application of F, either it's a fixed-point, or the result must go down at least one level in lattice
- eventually must hit a fixed-point (which will be the best fixed-point) or bottom (which is guaranteed to be a fixed-point), if *D* of finite height

Complexity of iterative analysis

How long does iterative analysis take?

`l`: depth of loop nesting
`n`: # of stmts in loop
`t`: time to execute one flow function
`k`: height of lattice

Another example: integer range analysis

For each program point,
for each integer-typed variable,
calculate (an approximation to) the set of integer values
that can be taken on by the variable

- use info for constant folding comparisons,
for eliminating array bounds checks,
for (in)dependence testing of array accesses,
for eliminating overflow checks

What domain to use?

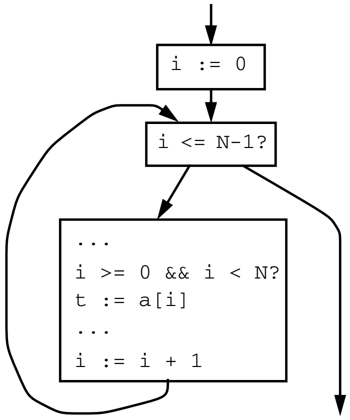
- what is its height?

What flow functions to use?

- are they monotonic?

Example

```
for i := 0 to N-1
  ... a[i] ...
end
```



```
i := 0
```

```
i <= N-1?
```

```
...
i >= 0 && i < N?
t := a[i]
...
i := i + 1
```

Widening operators

If domain is tall, then can introduce artificial generalizations (called **widenings**) when merging at loop heads

- ensure that only a finite number of widenings are possible
- not easy to design the “right” widening strategy

A generic worklist algorithm for lattice-theoretic DFA

Maintain a mapping from each program point to info at that point

- optimistically initialize all pp’s to T

Set initial pp’s (e.g. entry/exit point) to their correct values

Maintain a worklist of nodes whose flow functions need to be evaluated

- initialize with all nodes in graph
- include explicit meet (merge) & widening-meet (loop-head-merge) nodes

```
While worklist nonempty do
  Remove a node from worklist
  Evaluate the node’s flow function,
    given current info on predecessor(successor) pp’s,
    allowing it to change info on successor(predecessor) pp’s
  If any pp info changed, put successor(predecessor) nodes
    on worklist (if not already there)
```

For faster analysis, want to follow topological order

- number nodes in forward(backward) topological order
- remove nodes from worklist in increasing topological order