A Selective Overview of Graph Cut Energy Minimization Algorithms

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Outline

Philosophy and motivation

1. Graph cut algorithms
2. Using graph cuts for energy minimization in vision
3. What energy functions can be minimized via graph cuts?

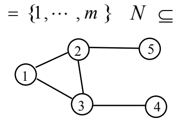
Problem reductions

- Suppose you're given a problem you don't know how to solve
  - Turn it into one that you <u>can</u> solve
- If any instance of a hard problem can be turned into your problem, then your problem is at least as hard

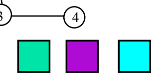
Pixel labeling problem

Given

$S = \{1, \cdots, m\}$ $\quad N \subseteq S \times S$



$L = \{l_1, \cdots, l_k\}$

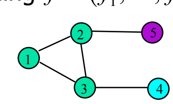


Assignment cost for giving a particular label to a particular node. Written as D.

Separation cost for assigning a particular pair of labels to neighboring nodes. Written as V.

Find

Labeling $f = (f_1, \cdots, f_m)$



Such that the sum of the assignment costs and separation costs (the energy E) is small

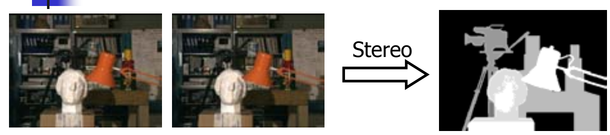
Solving pixel labeling problems

- We want to minimize the energy $E(f)$

$$\arg\min_{f} \underbrace{\sum_{p} D(p, f_p)}_{\text{assignment costs}} + \underbrace{\sum_{p,q \in N} V(f_p, f_q)}_{\text{separation costs}}$$

- Problem show up constantly in vision
  - And in other fields as well
- Bayesian justification (MRF's)

Pixel labeling for stereo

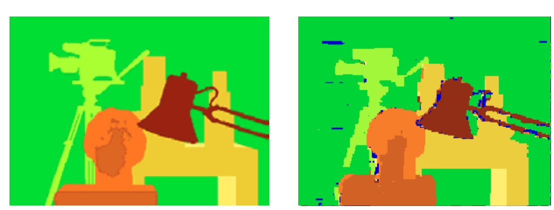


- Labels are shifts (disparities, hence depths)
- At the right disparity, $I_1(p) \cong I_2(p+d)$
  - Assignment cost is $D(p,d) = (I_1(p) - I_2(p+d))^2$
- Neighboring pixels should be at similar depths
  - Except at the borders of objects!

Classical solutions

- No good solutions until recently
- General purpose optimization methods
  - Simulated annealing, or some such
  - Bad answers, slowly
- Local methods
  - Each pixel chooses a label independently
  - Bad answers, fast

How fast do you want the wrong answer?



Right answers

Slow (annealing) / Graph cuts / Fast

What do graph cuts provide?

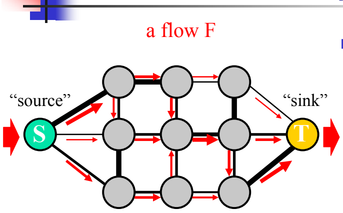
- For less interesting V, polynomial algorithm for global minimum!
- For a particularly interesting V, approximation algorithm
  - Proof of NP hardness
- For many choices of V, algorithms that find a strong local minimum
- Very strong experimental results

Part A: Graph Cuts

Everything you never wanted to know about cuts and flows

(but were afraid to ask)

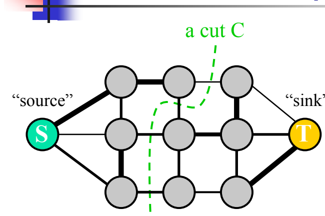
Maximum flow problem



A graph with two terminals

- Max flow problem:
  - Each edge is a "pipe"
  - Find the largest flow F of "water" that can be sent from the "source" to the "sink" along the pipes
  - Edge weights give the pipe's capacity

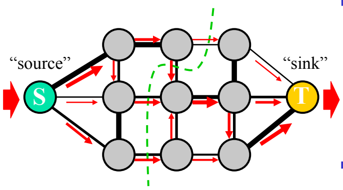
Minimum cut problem



A graph with two terminals

- Min cut problem:
  - Find the cheapest way to cut the edges so that the "source" is completely separated from the "sink"
  - Edge weights now represent cutting "costs"

Max flow/Min cut theorem



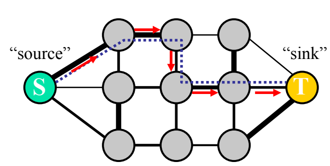
A graph with two terminals

- Max Flow = Min Cut:
  - Maximum flow saturates the edges along the minimum cut.
  - Ford and Fulkerson, 1962
  - Problem reduction!
- Ford and Fulkerson gave first polynomial time algorithm for globally optimal solution

Fast algorithms for min cut

- Max flow problem can be solved fast
  - Many algorithms, we'll sketch one
- This is not at all obvious
  - Variants of min cut are NP-hard
- Multiway cut problem
  - More than 2 terminals
  - Find lowest cost edges separating them all

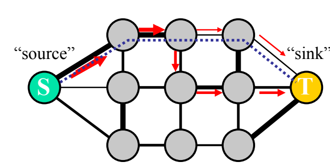
"Augmenting Path" algorithms



A graph with two terminals

- Find a path from S to T along non-saturated edges
- Increase flow along this path until some edge saturates

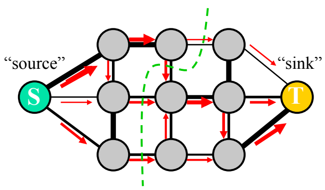
"Augmenting Path" algorithms



A graph with two terminals

- Find a path from S to T along non-saturated edges
- Increase flow along this path until some edge saturates
- Find next path…
- Increase flow…

“Augmenting Path” algorithms



A graph with two terminals

- Find a path from S to T along non-saturated edges
- Increase flow along this path until some edge saturates

Iterate until …
all paths from S to T have at least one saturated edge

Implementation notes

- There are many fast flow algorithms
- Augmenting paths depends on ordering
  - Breadth first = Edmonds-Karp
  - Vision problems have many short paths
  - Subtleties needed due to directed edges
- [BK ’04] gives an algorithm especially for vision problems
  - Software is freely available

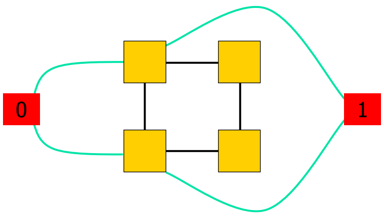
Part B: Graph cuts in vision

How you can turn vision problems
into min cut problems
for fun and profit (or, not)

A surprising result

- Minimizing E in vision is difficult
  - Huge search space
  - Many local minima
- With 2 labels, can find the global min!
  - [Greig, Porteus, Shehult, 1986]
- Problem reduction to min cut

Construction



- Exactly 1 green link is cut, for every pixel
  - Cuts are labelings
  - Two obvious encodings
- If two adjacent pixels end up linked to different terminals, the black link between them must be cut
  - Cost of cut is energy of labeling

Smoothness term matters

- V determines what the solution prefers
  - Consider uniform D
  - Comes from the MRF's prior
- Convex V over-penalizes discontinuities

  $|d1 - d2|$
- Non-convex V is important

  $T[d_1 != d_2]$, called the Potts model

  $min(|d1 - d2|, K)$

Why is the problem hard?

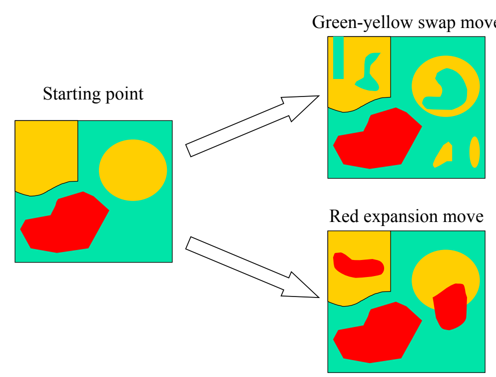
- Minimizing the energy for the interesting choices of V is NP-hard
  - Problem reduction from multiway cut
  - Result is somewhat recent [BV&Z '01]
- Requires exponential search
  - Dimension = number of pixels

Convex V is easy

- Graph cuts can rapidly compute the global minimum
  - Convex V, contiguous integer labels
  - [Ishikawa '04]
- Another amazing result
  - Not of practical interest (IMHO)
  - You can really see the over-smoothing

Local minima and moves

- A local minimum is all we can hope for
  - For the important class of V
- Minimum relative to a set of moves
  - Better than any "nearby" solution
- We will compute a local minimum with respect to very powerful moves



Swap move algorithm

- 1. Start with an arbitrary labeling
- 2. Cycle through every label pair (α,β) in some order
  - 2.1 Find the lowest E labeling within a single α,β-swap
  - 2.2 Go there if this has lower E than the current labeling
- 3. If E did not decrease in the cycle, we're done. Otherwise, go to step 2

Expansion move algorithm

- 1. Start with an arbitrary labeling
- 2. For each label α in some order
  - 2.1 Find the lowest E labeling within a single α-expansion
  - 2.2 Go there if this has lower E than the current labeling
- 3. If E did not decrease in the cycle, we're done. Otherwise, go to step 2

Algorithm properties

- In a cycle the energy E doesn't increase
  - These are greedy methods
  - Convergence guaranteed in $O(n)$ cycles
    - In practice, termination occurs in a few cycles
- When the algorithms converge, the resulting labeling is a local minimum
  - Even when allowing an arbitrary swap move or expansion move

Strong local minima

- A local minimum with respect to these moves is the best answer in a very large neighborhood
  - For example, there are $O(k\ 2^n)$ labelings within a single expansion move
  - Starting at an arbitrary labeling, you can get to the global minimum in k expansion moves
- The expansion move algorithm yields a 2-approximation for Potts model V

Binary sub-problem

- The input problem involves k labels, but the key sub-problem involves only 2
  - Minimize E over all $O(2^n)$ labelings within a single α-expansion move from f
  - Each pixel p either keeps its old label f_p, or acquire the new label α
- Classical problem reduction
  - To min cut problem

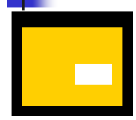
Part C: What energy functions can graph cuts minimize?

Or, what else can we do with this?

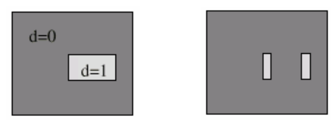
Different D example

- Birchfield-Tomasi method
  - Compute an intensity interval
    - Use this as basis for $D(p,d)$
  - Handles sampling error

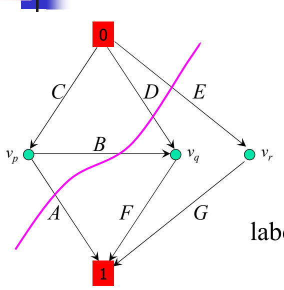
Different V example



Stereo Image: White rectangle moves one pixel, background is stationary



Cuts are binary labelings



$$S = \{0, v_p\}$$
$$T = \{1, v_q, v_r\}$$
$$\text{labeling} = \{v_p \leftarrow 0; v_q \leftarrow 1; v_r \leftarrow 1\}$$
$$\text{cost} = A + B + D + E$$

Recent progress

- Over the last 4 years several such problem reductions have been done
  - Graph constructions have been specialized to a particular E, and quite complex
  - You can't tell by looking at E whether such a construction is possible, let alone how to do it
- We now have a much more general result for energy functions of binary-valued variables
  - Graph cuts invariably use binary variables

Energy functions and graphs

- Consider E which assigns to any cut (binary labeling) the cost of that cut
  - Computing the min cut on G is a fast way to find the labeling that minimizes E
  - We will say that G represents E
  - Every weighted graph with two terminals represents some energy function
  - What we really want is to go in the opposite direction!

Basic questions

- For what energy functions E can we construct a graph G that represents E?
  - I.e., what energy functions can we efficiently minimize using graph cuts?
- How can we easily construct the graph G that represents E?
  - I.e., given an arbitrary E that we know to be graph-representable, how do we find G?

Question #1

What class of energy functions can we efficiently minimize using graph cuts?

The classes F2 and F3

- Consider functions of n binary variables
- Functions in F2 can be written as

$$\sum_i E^i(v_i) + \sum_{i<j} E^{i,j}(v_i, v_j)$$

- Functions in F3 can be written as

$$\sum_i E^i(v_i) + \sum_{i<j} E^{i,j}(v_i, v_j) + \sum_{i<j<k} E^{i,j,k}(v_i, v_j, v_k)$$

Regularity

- All functions E of 1 binary variable are defined to be regular
- A function E of 2 binary variables is regular if

$$E(0,0) + E(1,1) \leq E(1,0) + E(0,1)$$

- A function E of more than 2 binary variables is regular if all its projections are regular

Regularity theorem

- A graph-representable function of binary variables must be regular
- In fact, minimizing an arbitrary non-regular functions in F2 is NP-hard
  - Reduction from independent set problem

F3 Theorem

- Any regular function in F3 is graph-representable
- With the regularity theorem, this completely characterizes the energy functions E that can be efficiently minimized with graph cuts
  - Assuming E has no terms with >3 variables

Question #2

Given: an arbitrary graph-representable E

Question: How do we find G ?

Desired construction

- Input: an arbitrary regular $E \in$ F3
- Output: the graph that represents E

Additivity theorem

- The sum of two graph-representable energy functions is itself graph-representable
  - Assume that the two graphs are defined on the same sets of vertices, i.e. they differ only in the edge weights
  - We simply add the edge weights together
    - If there is no edge, we can pretend there is one with weight 0

Regrouping theorem

- Any regular function in F3 can be re-written as the sum of terms, where each term is regular and involves 3 or fewer variables
  - Combined with the additivity theorem, we need only build a graph for an arbitrary regular term involving 3 or fewer variables

Construction for F2

- Consider an arbitrary regular E in F2
- We only need to look at a single term, whose form is like $D(v_p)$ or $V(v_p, v_q)$
  - Example: expansion moves for stereo
  - We will show how the construction works for both types of terms
  - Each term is known to be regular!

Data terms

- We need a graph construction that imposes one "penalty" if $v_p = 0$, but another if $v_p = 1$
  - The "penalty" can be positive or negative!

$$\begin{array}{c} {\scriptstyle v_p=0} \\ {\scriptstyle v_p=1} \end{array} \begin{array}{|c|} \hline D(p,f_p) \\ \hline D(p,\alpha) \\ \hline \end{array} = \begin{array}{|c|} \hline E(0) \\ \hline E(1) \\ \hline \end{array}$$

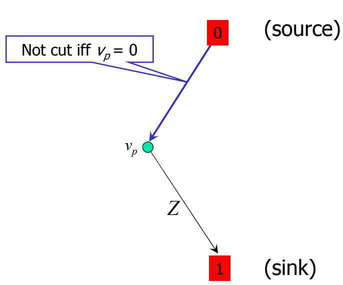
Rewriting the penalty

$$\begin{array}{c} {\scriptstyle v_p=0} \\ {\scriptstyle v_p=1} \end{array} \begin{array}{|c|} \hline E(0) \\ \hline E(1) \\ \hline \end{array} = \begin{array}{|c|} \hline E(1) \\ \hline E(1) \\ \hline \end{array} + \begin{array}{|c|} \hline E(0)-E(1) \\ \hline 0 \\ \hline \end{array}$$

- The penalty $Z = E(0) - E(1)$, imposed iff $v_p = 0$, can be positive or negative
  - But graph has non-negative edge weights!

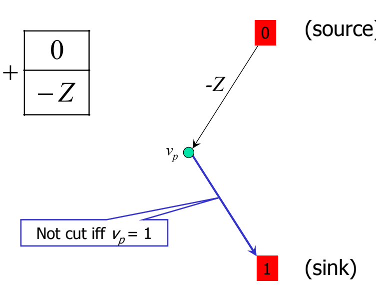
Case 1: $Z > 0$

$$\begin{array}{c} {\scriptstyle v_p=0} \\ {\scriptstyle v_p=1} \end{array} \begin{array}{|c|} \hline Z \\ \hline 0 \\ \hline \end{array}$$



Case 2: $Z < 0$

$$\begin{array}{c} {\scriptstyle v_p=0} \\ {\scriptstyle v_p=1} \end{array} \begin{array}{|c|} \hline Z \\ \hline 0 \\ \hline \end{array} = \begin{array}{|c|} \hline Z \\ \hline Z \\ \hline \end{array} + \begin{array}{|c|} \hline 0 \\ \hline -Z \\ \hline \end{array}$$



Smoothness terms

	$v_q = 0$	$v_q = 1$			
$v_p = 0$	$V(f_p, f_q)$	$V(f_p, \alpha)$	$=$	A	B
$v_p = 1$	$V(\alpha, f_q)$	$V(\alpha, \alpha)$		C	D

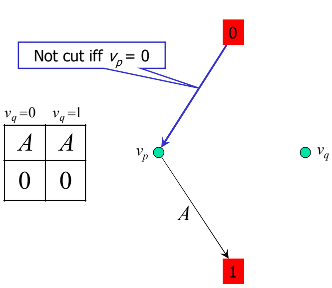
- We will assume $A>0$, $C>0$, $D\text{-}C>0$ and construct the appropriate graph
  - The other cases are very similar

Rewriting the term

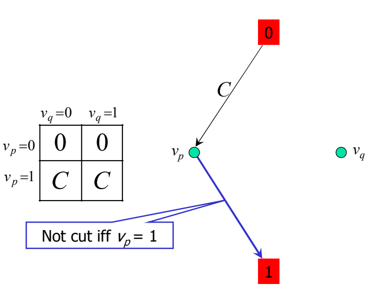
$$\begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline\end{array} = \begin{array}{|c|c|}\hline A & A \\ \hline 0 & 0 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline C & C \\ \hline\end{array} + \begin{array}{|c|c|}\hline 0 & D-C \\ \hline 0 & D-C \\ \hline\end{array} + \begin{array}{|c|c|}\hline 0 & B+C-A-D \\ \hline 0 & 0 \\ \hline\end{array}$$

- All the entries are non-negative by assumption except the last
  - Which is non-negative by regularity!

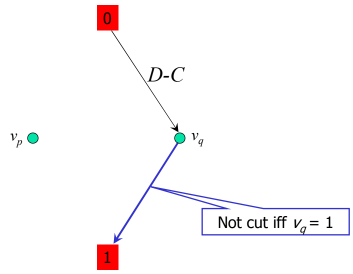
	$v_q = 0$	$v_q = 1$
$v_p = 0$	A	A
$v_p = 1$	0	0



	$v_q = 0$	$v_q = 1$
$v_p = 0$	0	0
$v_p = 1$	C	C

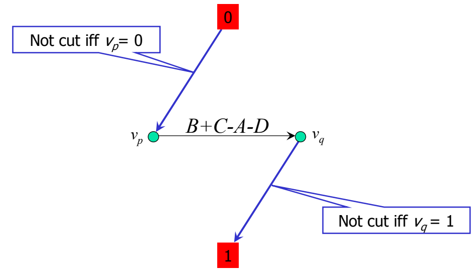


	$v_q=0$	$v_q=1$
$v_p=0$	0	$D-C$
$v_p=1$	0	$D-C$



	$v_q=0$	$v_q=1$
$v_p=0$	0	$B+C-A-D$
$v_p=1$	0	0

Non-negative by regularity

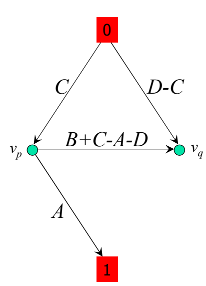


Putting it all together

	$v_q=0$	$v_q=1$
$v_p=0$	A	B
$v_p=1$	C	D

$A > 0, C > 0$

$D - C > 0$

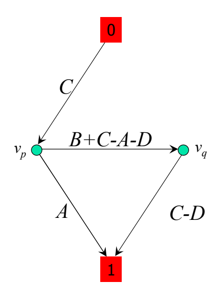


One of the other cases

	$v_q=0$	$v_q=1$
$v_p=0$	A	B
$v_p=1$	C	D

$A > 0, C > 0$

$\boxed{D - C < 0}$



What energy functions?

- Partial characterization of the energy functions of binary variables that can be minimized with graph cuts
- General-purpose construction for an arbitrary regular function in F3
- It is no longer necessary to explicitly build the graph (do the problem reduction)
  - Instead, simply check the regularity condition and apply our construction

Conclusions

- Problem reductions are powerful
  - Beware of general-purpose solutions!
  - Special-purpose ones are fragile
- Graph cuts have both theoretical and practical interest
- They are now (relatively) easy to use