$$
\begin{aligned}
v_* \in V &::= n \mid (\lambda x.M) \\
e_* \in M &::= V \\
&\mid (op\ M\ M) \qquad ;\ op \in \{+,-,*,/,\ldots\} \\
&\mid (\text{if0}\ M\ M\ M) \\
&\mid (M\ M) \\
&\mid x
\end{aligned}
$$

$$n \Downarrow n \quad [\text{num}]$$

$$(\lambda x.e) \Downarrow (\lambda x.e) \quad [\lambda]$$

$$\frac{e_1 \Downarrow n_1 \quad e_2 \Downarrow n_2 \quad \ulcorner op \urcorner(n_1, n_2) = v}{(op\ e_1\ e_2) \Downarrow v}\ [op]$$

$$\frac{e_c \Downarrow 0 \quad e_t \Downarrow v}{(\text{if0}\ e_c\ e_t\ e_f) \Downarrow v}\ [\text{if0-t}]$$

$$\frac{e_c \Downarrow n\ (n \neq 0) \quad e_f \Downarrow v}{(\text{if0}\ e_c\ e_t\ e_f) \Downarrow v}\ [\text{if0-f}]$$

An example:

$$\dfrac{(\lambda y.(*\ y\ y)) \Downarrow \text{itself} \quad \dfrac{\dfrac{1 \Downarrow 1 \quad 2 \Downarrow 2 \quad \ulcorner + \urcorner(1,2) = 3}{(+\ 1\ 2) \Downarrow 3} \quad 3 \neq 0 \quad 4 \Downarrow 4}{(\text{if0}\ (+\ 1\ 2)\ \Omega\ 4) \Downarrow 4} \quad \dfrac{4 \Downarrow 4 \quad \ulcorner * \urcorner(4,4) = 16}{(*\ y\ y)[y \leftarrow 4] = (*\ 4\ 4) \Downarrow 16}}{((\lambda y.(*\ y\ y))\ (\text{if0}\ (+\ 1\ 2)\ \Omega\ 4)) \Downarrow 16}$$

$$\frac{e_f \Downarrow (\lambda x.e) \quad e_a \Downarrow v_a \quad e[x \leftarrow v_a] \Downarrow v}{(e_f\ e_a) \Downarrow v}\ [\beta_v]$$

$$\frac{e_f \Downarrow (\lambda x.e) \quad e[x \leftarrow e_a] \Downarrow v}{(e_f\ e_a) \Downarrow v}\ [\beta_n]$$

Add mutation: $(\text{set!}\ x\ M)$

$$\frac{e_f ; \Sigma \Downarrow (\lambda x.e) ; \Sigma_1 \quad e_a ; \Sigma_1 \Downarrow v_a ; \Sigma_2 \quad \sigma \notin dom(\Sigma_2) \quad e[x \leftarrow \sigma] ; \Sigma_2[\sigma \mapsto v_a] \Downarrow v ; \Sigma_3}{(e_f\ e_a) ; \Sigma \Downarrow v ; \Sigma_3}\ [\beta_v\text{-}\Sigma]$$

$$\sigma ; \Sigma \Downarrow \Sigma(\sigma) ; \Sigma \quad [\sigma]$$

$$\frac{e ; \Sigma \Downarrow v ; \Sigma_1}{(\text{set!}\ \sigma\ e) ; \Sigma \Downarrow v ; \Sigma_1[\sigma \mapsto v]}\ [\text{set!}]$$

(and extend other rules similarly)

$$
\begin{aligned}
E ::= &\ [\,] \\
\mid &\ (op\ E\ M) \\
\mid &\ (op\ V\ E) \\
\mid &\ (\text{if}0\ E\ M\ M) \\
\mid &\ (E\ M) \\
\mid &\ ((\lambda x.M)\ E)
\end{aligned}
$$

Modeling non-local control:

$$[\text{call/cc}]\quad E[(\text{call/cc}\ (\lambda k.e))] \to E[e[k \leftarrow (\hat{\lambda}.E)]]$$

$$[k]\quad E[((\hat{\lambda}.E')\ v)] \to E'[v]$$

Small steps:

$$E[(op\ n_1\ n_2)] \to E[n] \text{ where } n = \ulcorner op \urcorner(n_1, n_2)$$

$$E[(\text{if}0\ 0\ e_1\ e_2)] \to E[e_1]$$

$$E[(\text{if}0\ n\ e_1\ e_2)] \to E[e_2] \text{ if } n \neq 0$$

$$E[((\lambda x.e)\ v)] \to E[e[x \leftarrow v]]$$

Adding mutation:

$$E[((\lambda x.e)\ v)]; \Sigma \to E[e[x \leftarrow \sigma]]; \Sigma[\sigma \mapsto v] \quad (\sigma \notin dom(\Sigma))$$

$$E[(\text{set!}\ \sigma\ v)]; \Sigma \to E[v]; \Sigma[\sigma \mapsto v]$$

(Σ passes through unchanged in extended forms of other rules.)

Example:

$$e = ((\lambda y.(*\ y\ y))\ (\text{if}0\ [(+\ 1\ 2)]\ \Omega\ 4))\ ;\ E = ((\lambda y.(*\ y\ y))\ (\text{if}0\ [\,]\ \Omega\ 4))$$

$$E[(+\ 1\ 2)] \to E[3] = ((\lambda y.(*\ y\ y))\ [(\text{if}0\ 3\ \Omega\ 4)]);\ E = ((\lambda y.(*\ y\ y))\ [\,])$$

$$E[(\text{if}0\ 3\ \Omega\ 4)] \to E[4] = [((\lambda y.(*\ y\ y))\ 4)];\ E = [\,]$$

$$E[((\lambda y.(*\ y\ y))\ 4] \to E[(*\ y\ y)[y \leftarrow 4]] = E[(*\ 4\ 4)] = [(*\ 4\ 4)];\ E = [\,]$$

$$E[(*\ 4\ 4)] \to E[\ulcorner * \urcorner(4,4)] = E[16] = 16$$