CSEP 505: Programming Languages

Lecture 7
February 19, 2015

v ::= n | **true** | **false** | *op* | **λ***x*.e e ::= v | **if** e e e | *x* | e e

$$v \Downarrow v \qquad \text{(VAL)}$$

$$\frac{e_c \Downarrow \text{true} \qquad e_t \Downarrow v}{(\text{if } e_c\ e_t\ e_f) \Downarrow v} \qquad \text{(IF-TRUE)}$$

$$\frac{e_c \Downarrow \text{false} \qquad e_f \Downarrow v}{(\text{if } e_c\ e_t\ e_f) \Downarrow v} \qquad \text{(IF-FALSE)}$$

$$\frac{e_f \Downarrow op \qquad e_a \Downarrow v_a \qquad \delta(op, v_a) = v}{(e_f\ e_a) \Downarrow v} \qquad (\delta)$$

$$\frac{e_f \Downarrow (\lambda x.e_b) \qquad e_a \Downarrow v_a \qquad e_b[x \leftarrow v_a] \Downarrow v}{(e_f\ e_a) \Downarrow v} \qquad (\beta_v)$$

v ::= n | **true** | **false** | *op* | **λ***x*.e   e ::= v | **if** e e e | *x* | e e

$$\dfrac{(\lambda x.x\ x) \Downarrow (\lambda x.x\ x) \qquad (\lambda x.x\ x) \Downarrow (\lambda x.x\ x) \qquad \dfrac{\vdots}{((\lambda x.x\ x)\ (\lambda x.x\ x)) \Downarrow \ldots}}{((\lambda x.x\ x)\ (\lambda x.x\ x)) \Downarrow \ldots}$$

$$\dfrac{\dfrac{(\lambda x.\textbf{if}\ \ldots) \Downarrow \cdot \qquad 3 \Downarrow 3 \qquad \dfrac{\dfrac{\delta(\text{iszero}, 3) = \text{false}}{\text{iszero } 3 \Downarrow \text{false}} \qquad \dfrac{\delta(\text{add}, 3) \Downarrow (3+)}{\text{add } 3 \Downarrow (3+)}}{\textbf{if}\ (\text{iszero } 3)\ \text{succ}\ (\text{add } 3) \Downarrow (3+)}}{((\lambda x.\textbf{if}\ \ldots)\ 3) \Downarrow (3+)} \qquad 4 \Downarrow 4 \qquad \delta((3+), 4) = 7}{(((\lambda x.\textbf{if}\ (\text{iszero } x)\ \text{succ}\ (\text{add } x))\ 3)\ 4) \Downarrow 7}$$

v ::= n | **true** | **false** | *op* | **λ***x*.e

e ::= v | **if** e e e | *x* | e e

$$\mathbf{if\ true\ e_t\ e_f} \rightarrow \mathbf{e_t} \qquad [\mathbf{if\text{-}t}]$$

$$\mathbf{if\ false\ e_t\ e_f} \rightarrow \mathbf{e_f} \qquad [\mathbf{if\text{-}f}]$$

$$\mathbf{op\ v} \rightarrow \boldsymbol{\delta}\mathbf{(op,\ v)} \qquad [\boldsymbol{\delta}]$$

$$\mathbf{(\lambda x.e)\ v} \rightarrow \mathbf{[x \leftarrow v]b} \qquad [\beta_v]$$

v ::= n | **true** | **false** | *op* | **λ***x*.e

e ::= v | **if** e e e | *x* | e e

$$\frac{e_f \to e'_f}{e_f\ e_a \to e'_f\ e_a} \quad \text{(APP-F)}$$

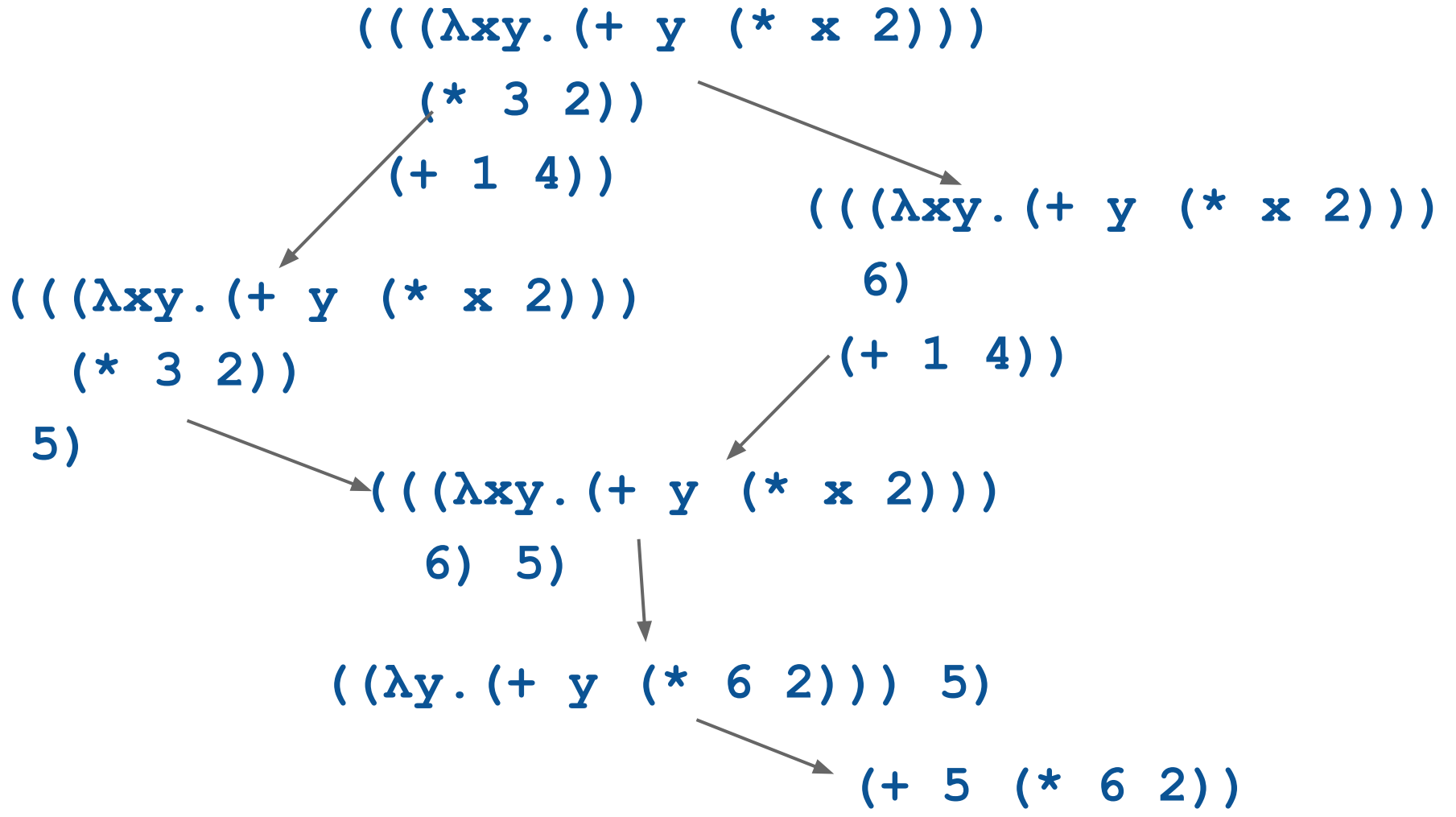
$$\frac{e_a \to e'_a}{e_f\ e_a \to e_f\ e'_a} \quad \text{(APP-A)}$$

$$\frac{e_c \to e'_c}{\textbf{if}\ e_c\ e_t\ e_f \to \textbf{if}\ e'_c\ e_t\ e_f} \quad \text{(IF-C)}$$

$$\frac{e_t \to e'_t}{\textbf{if}\ e_c\ e_t\ e_f \to \textbf{if}\ e_c\ e'_t\ e_f} \quad \text{(IF-T)}$$

$$\frac{e_f \to e'_f}{\textbf{if}\ e_c\ e_t\ e_f \to \textbf{if}\ e_c\ e_t\ e'_f} \quad \text{(IF-F)}$$

$$\frac{e \to e'}{\lambda x.e \to \lambda x.e'} \quad (\lambda)$$


(((λxy.(+ y (* x 2)))
(* 3 2))
(+ 1 4))
(((λxy.(+ y (* x 2)))
6)
(+ 1 4))
(((λxy.(+ y (* x 2)))
(* 3 2))
5)
(((λxy.(+ y (* x 2)))
6) 5)
((λy.(+ y (* 6 2))) 5)
(+ 5 (* 6 2))

v ::= n | **true** | **false** | *op* | **λ***x*.e

e ::= v | **if** e e e | *x* | e e

$$\frac{e_f \to e'_f}{e_f\ e_a \to e'_f\ e_a} \quad \text{(APP-F)}$$

$$\frac{e_a \to e'_a}{v\ e_a \to v\ e'_a} \quad \text{(APP-A)}$$

$$\frac{e_c \to e'_c}{\textbf{if}\ e_c\ e_t\ e_f \to \textbf{if}\ e'_c\ e_t\ e_f} \quad \text{(IF-C)}$$

$$\frac{e_t \to e'_t}{\textbf{if}\ e_c\ e_t\ e_f \to \textbf{if}\ e_c\ e'_t\ e_f} \quad \text{(IF-T)}$$

$$\frac{e_f \to e'_f}{\textbf{if}\ e_c\ e_t\ e_f \to \textbf{if}\ e_c\ e_t\ e'_f} \quad \text{(IF-F)}$$

$$\frac{e \to e'}{\lambda x.e \to \lambda x.e'} \quad (\lambda)$$

v ::= n | **true** | **false** | *op* | **λ***x*.e

e ::= v | **if** e e e | *x* | e e

$$\dfrac{\dfrac{\dfrac{\dfrac{\delta(+,2) = (2+)}{(+\ 2) \to (2+)}}{(+\ 2\ 3) \to ((2+)\ 3)}}{((\lambda x.\textbf{if}\ (\text{iszero}\ x)\ \text{succ}\ (\text{add}\ x))\ (+\ 2\ 3)) \to ((\lambda x.\textbf{if}\ (\text{iszero}\ x)\ \text{succ}\ (\text{add}\ x))\ ((2+)\ 3))}}{(((\lambda x.\textbf{if}\ (\text{iszero}\ x)\ \text{succ}\ (\text{add}\ x))\ (+\ 2\ 3))\ 4) \to (((\lambda x.\textbf{if}\ (\text{iszero}\ x)\ \text{succ}\ (\text{add}\ x))\ ((2+)\ 3))\ 4)}$$

v ::= n | **true** | **false** | *op* | **λ***x*.e

e ::= v | **if** e e e | *x* | e e

$$\frac{\dfrac{\dfrac{\delta((2+),3) = 5}{((2+)\ 3) \to 5}}{((\lambda x.\textbf{if}\ (\text{iszero}\ x)\ \text{succ}\ (\text{add}\ x))\ ((2+)\ 3)) \to ((\lambda x.\textbf{if}\ (\text{iszero}\ x)\ \text{succ}\ (\text{add}\ x))\ 5)}}{(((\lambda x.\textbf{if}\ (\text{iszero}\ x)\ \text{succ}\ (\text{add}\ x))\ ((2+)\ 3))\ 4) \to (((\lambda x.\textbf{if}\ (\text{iszero}\ x)\ \text{succ}\ (\text{add}\ x))\ 5)\ 4)}$$

v ::= n | **true** | **false** | *op* | **λ***x*.e e ::= v | **if** e e e | *x* | e e

$$\frac{((\lambda x.\textbf{if } (\text{iszero } x) \text{ succ } (\text{add } x))\ 5) \rightarrow (\textbf{if } (\text{iszero } 5) \text{ succ } (\text{add } 5))}{(((\lambda x.\textbf{if } (\text{iszero } x) \text{ succ } (\text{add } x))\ 5)\ 4) \rightarrow ((\textbf{if } (\text{iszero } 5) \text{ succ } (\text{add } 5))\ 4)}$$

$$\frac{\dfrac{\dfrac{\delta(\text{iszero}, 5) = \text{false}}{(\text{iszero } 5) \rightarrow \text{false}}}{(\textbf{if } (\text{iszero } 5) \text{ succ } (\text{add } 5)) \rightarrow (\textbf{if false succ } (\text{add } 5))}}{((\textbf{if } (\text{iszero } 5) \text{ succ } (\text{add } 5))\ 4) \rightarrow ((\textbf{if false succ } (\text{add } 5))\ 4)}$$

$$\frac{(\textbf{if false succ } (\text{add } 5)) \rightarrow (\text{add } 5)}{((\textbf{if false succ } (\text{add } 5))\ 4) \rightarrow ((\text{add } 5)\ 4)}$$

v ::= n | **true** | **false** | *op* | **λ***x*.e

e ::= v | **if** e e e | *x* | e e

```
(((λx.if (not x) succ (add x)) true) 3) →
((if (not true) succ (add true)) 3) →
((if false succ (add true)) 3) →
((add true) 3)
```

v ::= n | **true** | **false** | *op* | **λ***x*.e

e ::= v | **if** e e e | *x* | e e

```
((λx.x x) (λx.x x)) →
((λx.x x) (λx.x x)) →
((λx.x x) (λx.x x)) → ...
```

v ::= n | **true** | **false** | *op* | **λ***x*.e e ::= v | **if** e e e | *x* | e e

C ::= [ ] | if C e e | if e C e | if e e C | C e | e C

$$\mathbf{C[if\ true\ e_t\ e_f]} \rightarrow \mathbf{C[e_t]} \qquad \mathbf{[if\text{-}t]}$$

$$\mathbf{C[if\ false\ e_t\ e_f]} \rightarrow \mathbf{C[e_f]} \qquad \mathbf{[if\text{-}f]}$$

$$\mathbf{C[op\ v]} \rightarrow \mathbf{C[}\delta\mathbf{(op,\ v)]} \qquad \mathbf{[}\delta\mathbf{]}$$

$$\mathbf{C[(\lambda x.e)\ v]} \rightarrow \mathbf{C[[x \leftarrow v]b]} \qquad \mathbf{[}\beta_v\mathbf{]}$$

v ::= n | **true** | **false** | *op* | **λ***x*.e

e ::= v | **if** e e e | *x* | e e

C ::= [ ] | if C e e | if e C e | if e e C | C e | e C

E ::= [ ] | if E e e | E e | v E

$$\mathbf{E}[\texttt{if true } e_t\ e_f] \rightarrow \mathbf{E}[e_t] \qquad [\texttt{if-t}]$$

$$\mathbf{E}[\texttt{if false } e_t\ e_f] \rightarrow \mathbf{E}[e_f] \qquad [\texttt{if-f}]$$

$$\mathbf{E}[\texttt{op v}] \rightarrow \mathbf{E}[\delta(\texttt{op, v})] \qquad [\delta]$$

$$\mathbf{E}[(\lambda \texttt{x.e})\ \texttt{v}] \rightarrow \mathbf{E}[[\texttt{x} \leftarrow \texttt{v}]\texttt{b}] \qquad [\beta_v]$$

v ::= n | **true** | **false** | *op* | **λ***x*.e e ::= v | **if** e e e | *x* | e e

C ::= [ ] | if C e e | if e C e | if e e C | C e | e C

E ::= [ ] | if E e e | E e | v E

$$\mathbf{E}[\texttt{if true } e_t\ e_f] \rightarrow \mathbf{E}[e_t] \qquad [\texttt{if-t}]$$

$$\mathbf{E}[\texttt{if false } e_t\ e_f] \rightarrow \mathbf{E}[e_f] \qquad [\texttt{if-f}]$$

$$\mathbf{E}[\texttt{op v}] \rightarrow \mathbf{E}[\delta(\texttt{op, v})] \qquad [\delta]$$

$$\mathbf{E}[(\lambda \texttt{x.e})\ \texttt{v}] \rightarrow \mathbf{E}[[\texttt{x} \leftarrow \texttt{v}]\texttt{b}] \qquad [\beta_v]$$

$$\mathbf{E}[\texttt{let/cc k e}] \rightarrow \mathbf{E}[[\texttt{k} \leftarrow \lambda_k.\mathbf{E}]\texttt{e}] \qquad [\texttt{let/cc}]$$

$$\mathbf{E}[(\lambda_k.\mathbf{E}')\ \texttt{v}] \rightarrow \mathbf{E}'[\texttt{v}] \qquad [\texttt{cont}]$$

(let/cc k (not (iszero (succ (k 3))))) →

(not (iszero (succ ((λ$_k$.[]) 3)))) →

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Typing Examples

```
(+ 1 (* 2 3))
```

```
(+ 1 (* 2 false))
```

```
(if (not true) 3 7)
```

```
(if (succ 0) 3 7)
```

```
(if false empty 7)
```

```
(if (empty? x)
    x
    (cons 5 empty))
```

```
(fun (x)
  (if x
      succ
      (+ x)))
```

```
(cons 1 (cons 2 empty))
```

```
(cons 1 (cons true empty))
```

```
(fun (f g x)
  (cons (f x)
    (cons (g x) empty)
```

```
(pair 1 2)
```

```
(pair 1 false)
```

```
(pair empty false)
```

```
(if …
    (pair empty (cons 1 empty))
    (pair (cons 3 empty) empty))
```

```
(if …
    (pair empty (cons 1 empty))
    (pair (cons true empty) empty))
```

```
((fix (fun (f g a b)
   (cons a
      (if (empty? b)
          empty
          (f g (f a (first b))
               (rest b))))))
 mult 1)
```

```
(fun (f g x)
  (pair (f (g x))
        ((f g) x)))
```

v ::= n | **true** | **false** | *op* | **λ***x*.e

e ::= v | **if** e e e | *x* | e e | let x = e in e

t ::= num | bool | t → t | [t] | t x t

$$\Gamma \vdash x : \Gamma(x) \qquad \text{(VAR)}$$

$$\Gamma \vdash n : \text{num} \qquad \text{(NUM)}$$

$$\frac{\Gamma \vdash e_c : \text{bool} \qquad \Gamma \vdash e_t : \tau \qquad \Gamma \vdash e_f : \tau}{\Gamma \vdash (\text{if } e_c \ e_t \ e_f) : \tau} \qquad \text{(IF)}$$

$$\frac{\Gamma \vdash e : \tau_x \qquad \Gamma, x : \tau_x \vdash e_b : \tau}{\Gamma \vdash (\text{let } x = e \text{ in } e_b) : \tau} \qquad \text{(LET)}$$

$$\frac{\Gamma, x : \tau_a \vdash e : \tau}{\Gamma \vdash (\lambda x : \tau_a . e) : \tau_a \to \tau} \qquad (\lambda)$$

$$\frac{\Gamma \vdash e_f : \tau_a \to \tau \qquad \Gamma \vdash e_a : \tau_a}{\Gamma \vdash (e_f \ e_a) : \tau} \qquad \text{(APP)}$$